The lexicographic cross-section of the plactic monoid is regular

Christian Choffrut[1] and Robert Mercaş[2]*

[1] L.I.A.F.A., Université Paris 7, 2 Pl. Jussieu, 75 251 Paris Cedex, France,
Christian.Choffrut@liafa.univ-paris-diderot.fr
[2] Christian-Albrechts-Universität zu Kiel, Institut für Informatik,
D-24098 Kiel, Germany, rgm@informatik.uni-kiel.de

Abstract. The plactic monoid is the quotient of the free monoid by the congruence generated by Knuth's well-celebrated rules. It is well-known that the set of Young tableaux is a cross-section of this congruence which happens to be regular. The main result of this work shows that the set of alphabetically minimal elements in the congruence classes is also regular. We give a full combinatorial characterization of these minimal elements and show that constructing them is as fast as constructing a tableau.

1 Introduction

Young tableaux were introduced in 1900 as combinatorial objects for studying the linear representations of the symmetric group. They can be thought of as Ferrers diagrams filled with the n first nonnegative integers subject to ordering properties along the rows and columns. Allowing arbitrary repetitions of the same integer lead to more general objects, the so-called *semistandard Young tableaux*. Knuth considered them as a possible data structure for sorting but showed that they perform relatively poorly, cf. [10, paragraph 5.1.4.]. Here, we view Young tableaux as representatives of elements of a monoid, called the *plactic monoid* by Lascoux and Schützenberger. The purpose of this work is to study the probably most natural cross-section of this monoid, namely the set of lexicographically minimal elements of each class and to show that this set is regular, i.e., recognizable by a finite automaton.

When a monoid is specified by generators and relators, it is desirable, but not always possible, to have at one's disposal a regular set of representatives. A natural way of selecting a particular element in a congruence class is to pick up the lexicographically minimal element when it exists which is guaranteed when the classes are finite. Examples of such monoids are the trace monoids defined as the quotient of the free monoid by commutation relations of some pairs of the generators, see the classical textbooks [5, 11]. For these monoids, there exist two known normal forms of congruence classes. The first one is the Cartier-Foata normal form consisting of the products of successive ordered subalphabets that are allowed to occur in the word, [3]. A second normal form is simply defined as

* Supported by the Alexander von Humboldt Foundation and DFG grant 582014

the lexicographically minimal element of the class for an arbitrary ordering of the alphabets, [1]. In both cases, the set of representatives is regular. A stronger result would be that *all* regular subsets of the monoid, not only the monoid itself, have a regular cross-section, possibly composed of the minimal representatives. It can be shown that no regular cross-section exists for general trace monoids, which is equivalent to saying that regular subsets of trace monoids are not unambiguous in general.

Extensions of trace monoids considering partial commutations depending on the context were studied in [2]. This paper studies the plactic monoid generated by two elements and, shows that for all regular subsets of this monoid there exists a regular cross-section. In [4] the authors consider "half" of the rules of the plactic monoid and study the closure properties of the regular subsets which possess a regular cross-section.

In the most favorable case, not only the monoid has a regular set of representatives but also the multiplication by a generator, viewed as a binary relation, is recognized by a two tape finite automaton as developed in the theory of automatic groups, [8]. More generally, there is a vast literature on so-called automatic structures consisting of encoding the elements of an algebraic structure by words, and its operations by relations between words, in such a way that all these objects are recognizable by finite automata.

We briefly outline our contribution. In the preliminaries we recall all the material necessary for a good understanding of the results and we put the emphasis on the basic notions of plactic monoid, Young tableaux and so forth, for which we assume the reader has little familiarity. As much as possible we rely not only on formal definitions but also on illustrations through examples since the objects are of geometric nature.

The main result concerns the characterization of the lexicographically minimal words in a congruence class via the notion of P-sequence, see paragraph 3.1. This allows us to draw interesting conclusions such as the fact that the set of lexicographically minimal representatives is regular. As a byproduct we show a property "à la Green": the lexicographically minimal representative and the equivalent Young tableau have the same length distribution of their maximal columns. We also give an upper bound on the complexity for effectively computing this representative. In the last part we show via simple observations why the regularity of the lexicographic cross-section is remarkable: the plactic monoid is not "regularity" friendly. We end the paper with the solution of a problem concerning the relation of conjugacy as an illustration of the type of issues that we think still deserves investigation.

2 Preliminaries

We assume some familiarity of the reader with the first part of these preliminaries which deals with words and free monoids. We try to be more thoroughful in the second part and refer, among the vast literature, to [10] and [12, 13] for a more detailed exposition of the theory of the Young tableaux and the plactic monoid.

2.1 Words

Throughout this paper, we consider a finite alphabet Σ consisting of the first nonzero positive integers ordered in the usual way. The elements of the free monoid Σ^* generated by the alphabet are *words*, also called *strings*. The length of a word $u \in \Sigma^*$ is denoted by $|u|$. The *lexicographic ordering* on Σ^* is denoted by $<_{lex}$ and is defined by the condition $u <_{lex} v$ if u is a prefix of v or if $u = xau'$, $v = xbv'$ and $a < b$. Given a string $u \in \Sigma^*$, we denote by $\mathrm{H}(u)$ and $\mathrm{T}(u)$ respectively its first and last element (H for *head* and T for *tail*). E.g., $\mathrm{H}(2615) = 2$ and $\mathrm{T}(2615) = 5$. A (concatenation) product of n words $u_1, u_2, \ldots, u_n$ is simply written $u = u_1 u_2 \cdots u_n$. This also holds when the words u_i for $1 \leq i \leq n$ are themselves reduced to a single letter in which case $n = |u|$. This notation is a potential source of ambiguity which should be solved by the context. When we want to decompose each u_i into its letters, we use a double index: $u_i = u_{i,1} u_{i,2} \cdots u_{i,n_i}$, where $n_i = |u_i|$.

We are interested in two special types of words. A *column* is a word with strictly decreasing letters; a *row* is a word with nondecreasing letters (the choice of these terms is standard and justified by the notion of Young tableau, see below). Clearly, every word can be uniquely factored as a product of columns of maximal length (respectively as a product of rows of maximal length). E.g., with 314521 we have respectively three columns 31/4/521 and four rows 3/145/2/1.

Since this work is mainly interested in subsets of words, and more precisely in subsets which are computationally simple, we recall that $X \subseteq \Sigma^*$ is *regular* (or *recognizable*) if it can be recognized by a finite automaton. By Kleene theorem this is equivalent to saying that the subset is *rational*, i.e., that it can be constructed from the single letters by performing finitely many times one of the three operations of set union, set concatenation and Kleene star.

2.2 The plactic monoid

The *plactic monoid* is the quotient of the free monoid Σ^* by the congruence generated by the following relations, known as Knuth's rules

$$bac \equiv bca \text{ where } a < b \leq c,$$
$$acb \equiv cab \text{ where } a \leq b < c$$

The simplicity of the rules hides the complexity of the resulting monoid. In particular, it is clearly neither right nor left cancellative. Also, we do not know of any Knuth-Bendix method which would enable us to test equality of two elements of the monoid. Such a verification almost necessarily goes through the construction of the Young tableaux associated with the elements.

We recall the famous *bump rules* which are immediate application of the Knuth relations and on which the construction of the Young tableaux is based.

Lemma 1 (Bump rule for rows). *Let $u \in \Sigma^*$ be a row and let $a < T(u)$. Then $ua \equiv bxay$ where $xby = u$ and b is the leftmost element greater than a.* □

Example 2. $122\underline{3}45 \cdot 2 \equiv 3 \cdot 122\underline{2}45$

There is a similar rule for columns.

Lemma 3 (Bump rule for columns). *Let $u \in \Sigma^*$ be a column and let $a \leq H(u)$. Then $au \equiv xayb$ where $xby = u$ and b is the least element greater than or equal to a.* □

Example 4. $3 \cdot 54\underline{3}21 \equiv 54\underline{3}21 \cdot 3$, $2 \cdot 54\underline{3}1 \equiv 54\underline{2}1 \cdot 3$ and $5 \cdot \underline{5}4321 \equiv 54321 \cdot \underline{5}$.

2.3 Young tableaux

The definition of a Young tableau requires the following relation.

Definition 5. *A column u* dominates *a column v, written $u \succeq v$, if $|u| \geq |v|$ and if $u_{|u|-|v|+i} \leq v_i$ for all $i = 1, 2, \ldots, |v|$.*

This relation is clearly an ordering on the set of columns. There exists a graphical representation of a nonincreasing sequence of columns, namely $v_1 \succeq v_2 \succeq \ldots \succeq v_p$, called a *Young tableau.* Indeed, write each v_j vertically on the first quadrant of the discrete plane with the tail on the horizontal axis with each row left justified. Then each row of the tableau is a sequence of nondecreasing letters.

Example 6. A Young tableau

5
3 4 5
1 2 4 5

We recall Schensted's algorithm for associating a tableau $Y(u)$ with a word u. The tableau is constructed by reading off from left to right the letters of the word one at a time and by inserting them in the tableau under construction. Given the tableau for u, it suffices to show how to modify it in order to get the tableau for ua, $a \in \Sigma$. If a is greater than or equal to the rightmost letter of the bottom row, just append it to the right of this row. Otherwise, let b denote the element of the bottom row which is bumped out by a, as explained above. Substitute a for b and repeat the procedure by inserting b in the second lowest row of the Young tableau by applying the same rule, and so forth until reaching the top row, if necessary.

					← **5**	5
2 5		2 5	← **3**	2 3		2 3
1 2 3	← **2**	1 2 2		1 2 2		1 2 2
insert 2		insert 3		insert 5		done
bump out 3		bump out 5				

Due to the construction rules, it is clear that the tableau is congruent to the concatenation of its columns from left to right. It is also congruent to the concatenation of its rows from top to bottom.

In the sequel we use the same term “Young tableau” indifferently to denote the above diagram or the $\succeq$-sequence of columns, the context ensuring the notation is not ambiguous.

Dual to Young tableaux are contretableaux. The contretableau occupies the southwest quadrant of the plane. The rows are nonincreasing from right to left and the columns are strictly decreasing from top to bottom. The word is read off from right to left and the insertion rules are dual to those of the tableaux.

Example 7. A Young tableau and its equivalent contretableau

$$\begin{array}{lll} 5 & \qquad & 3\,5\,5 \\ 4\,5 & & 1\,4\,4 \\ 3\,3\,4 & & \,2\,3 \\ 1\,2\,2 & & \,2 \end{array}$$

The following technical result will be used in the proof of the main theorem.

Proposition 8. *Let u and v be two columns defining a Young tableau, i.e., $u \succeq v$ Let w be a column such that $\mathrm{H}(w) < \mathrm{T}(u)$. Then $uvw \equiv uwv$.*

Proof. Observe that it suffices to prove it in the case where w is a single letter a. The product uv is a Young tableau. Inserting a according to Schensted's rule yields the Young tableau uav. □

2.4 Cross-section

We recall that a *cross-section* of an equivalence relation is a set consisting of exactly one element in each class. It is known that Young tableaux, as well as contretableaux, define a cross-section of the plactic monoid, cf. [12, Thm 5.2.5.].

The purpose of this work is to prove that the cross-section of the lexicographically minimal representatives, abbreviated as lexicographic cross-section, is regular. For example it is an easy exercise to verify that $1^*(21)^*2^*$ is the lexicographic cross-section over the two letter alphabet. A bit more tedious is to verify that over a three letter alphabet the lexicographic cross-section is $1^*(21)^*(2^* + (31)^*)(321)^*(32)^*3^*$. The case of four letters can still be computed by hand and is again regular which led us to conjecture that this is a general result, but computing the five letter case is rather tedious. On the contrary, the set of lexicographically maximal representatives is not regular from the two-letter plactic monoid on, since it is the set $\{2^n 1^m 2^p \mid 0 \leq n < m \text{ if } p \neq 0\}$.

Clearly the set of Young tableaux provides us with the regular cross-section which is the finite union of all subsets of the form

$$v_1^* v_2^* \cdots v_p^* \quad \text{with } v_1 \succ v_2 \succ \ldots \succ v_p$$

This is a special case of the rational cross-section with entropy equal to 0, see [2, Proposition 8]. Actually, this subset lies very low in the hierarchy of rational subsets since it is *local* in the sense that the current state of any input depends on the last $|\Sigma| + 1$ letters. Still we believe it is challenging to ask whether the lexicographic cross-section is also regular.

3 Minimal representatives

3.1 A combinatorial property for minimal representatives

The following relation on the set of columns is needed for determining the lexicographically minimal representatives. We write $u \trianglelefteq v$ whenever the following conditions hold

- for all $i = 1, 2, \dots, \min\{|u|, |v|\}$, the condition $u_i \le v_i$ holds;
- furthermore, if $|u| < |v|$, then $u_{|u|} \le v_{|u|+1}$ holds.

The relation $\trianglelefteq$ is not transitive. Indeed, we have $432 \trianglelefteq 43$ and $43 \trianglelefteq 6541$ but the relation $432 \trianglelefteq 6541$ does not hold. However the transitive closure of the relation is an ordering.

Proposition 9. *The transitive closure of $\trianglelefteq$ is antisymmetric.*

Proof. We prove the result by contradiction. Assume there exists an element x and a sequence

$$z_0 \trianglelefteq z_1 \trianglelefteq \dots \trianglelefteq z_p \tag{1}$$

with $z_0 = z_p = x$, such that for some $0 < i < p$ we have $z_i \neq x$. Furthermore, we assume that p is minimal.

Let $0 < \mu \le p$ be the greatest integer i such that $|z_j| \ge |z_i|$, for all $0 \le j \le p$. If $\mu = p$, then for all $y = z_i$ and for all k with $0 \le k \le |x|$, by the definition of $\trianglelefteq$, we have $x_k \le y_k \le x_k$ and, thus, x is a prefix of y. Since x has a unique occurrence in the sequence, because of the minimality of its length, x is in particular a proper prefix of z_1, which violates the condition $z_0 \trianglelefteq z_1$. So we must assume $\mu < p$. Set $|z_\mu| = m$. Then, for all $1 \le i \le m$, we have

$$x_i \le z_{\mu,i} \le z_{\mu+1,i} \le x_i,$$

thus z_μ is a proper prefix of $z_{\mu+1}$, a contradiction of the relation $z_\mu \trianglelefteq z_{\mu+1}$. □

The characterization of the lexicographically minimal representatives is based on the following notion.

Definition 10. *A sequence of columns $u_1, u_2, \dots, u_n$ is a P-*sequence *if it satisfies the condition*

$$u_i \trianglelefteq u_j, \text{ for all } 1 \le i < j \le n. \tag{2}$$

The following technical result is crucial in establishing Theorem 12. The proof is routine and consists in applying inductively Proposition 8.

Lemma 11. *Let $u_1, u_2, \dots, u_n$ be a P-sequence such that $|u_1| \ge |u_i|$ for all $i = 2, \dots, n$. Then there exists a sequence of columns $w_1, w_2, \dots, w_n$ satisfying the following conditions*

(1) $|w_1| = |u_2|, |w_2| = |u_3|, \dots, |w_{n-1}| = |u_n|, |w_n| = |u_1|$

(2) *for all $1 \le k \le p$ we have $w_{n,k} = \max\{u_{i,k} \mid i = 1, 2, \dots, n\}$*

(3) *for all columns x such that $u_1 x$ is still a column we have*

$$u_1 x u_2 \cdots u_n \equiv w_1 w_2 \cdots w_n x$$

□

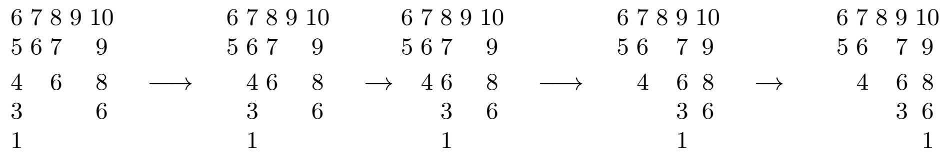

Fig. 1. An illustration of Lemma 11

Theorem 12. *Let $u = u_1 u_2 \cdots u_n$ be a factorization of maximal columns. Then, u is lexicographically minimal in its congruence class if and only if the sequence $u_1, u_2, \ldots, u_n$ is a* P*-sequence.*

Proof. That the condition is necessary in the case $n = 2$ is a consequence of the following.

Lemma 13. *Let u, v be two columns such that uv is not a column.*

(i) *if $v = v'w$ such that $|u| = |v'|$, $u \trianglelefteq v'$ and $H(w) < T(u)$, then $uv \equiv uwv'$.*

(ii) *if $u = u'xby$ and $v = v'a$ such that $|u'| = |v'|$, $u' \trianglelefteq v'$ and $b \leq a < T(x)$, then $uv \equiv u'bv'xay$ and $u'bv'xay \triangleleft uv$.*

Proof. Indeed, the first assertion is a consequence of Proposition 8. Concerning the second assertion we have

$$u'xbyv'a \equiv u'v'xbya \equiv u'v'bxay \equiv u'bv'xay$$

The first and third equivalences are obtained as an application of Proposition 8. The second one is an application of the bump rule on the column xby. □

We now return to the Theorem and prove the necessity of the condition. Let u be minimal in its class and assume by contradiction, that there exist i, j such that $u_i \ntrianglelefteq u_j$ with $i < j$ and $j - i$ minimal. By previous Lemma we have $j > i + 1$.

We first observe that all columns u_k with $i < k < j$ have length less than $\min\{|u_i|, |u_j|\}$. Indeed, consider first the case where there exists a minimal position p such that $u_{i,p} > u_{j,p}$, implying in particular $p \leq \min\{|u_i|, |u_j|\}$. Let u_k with $i < k < j$ be a column of length at least p. Then we have $u_{k,p} \geq u_{i,p} > u_{j,p}$, i.e., $u_k \ntrianglelefteq u_j$, contradicting the minimality of $j - i$. In the second case we have $u_{j,p+1} < u_{i,p}$, where $p = |u_i|$. Let u_k with $i < k < j$ be a column with length at least p. Since $u_k \trianglelefteq u_j$, we have either $u_{k,p+1} \leq u_{j,p+1}$, thus $u_{k,p+1} < u_{i,p}$, or $u_{k,p} \leq u_{j,p+1}$, thus $u_{k,p} < u_{i,p}$ whenever $|u_k| = p$, a contradiction in both cases with the minimality of $j - i$.

Now we shall consider the two causes for the condition $u_i \not\trianglelefteq u_j$ and we will show in both cases that it is possible to replace the factor $u_i u_{i+1} \cdots u_j$ by a factor of the same length but lexicographically smaller.

First we assume that there exists a minimal position p such that $u_{i,p} > u_{j,p}$. We set $u_i = u'_i x$, where u'_i is the prefix of u_i of length $p-1$. Applying Lemma 11 there exists a sequence of columns $w_1, w_2, \ldots, w_n$ such that

$$u'_i y u_{i+1} \cdots u_{j-1} \equiv w_i w_{i+1} \cdots w_{j-1} y \tag{3}$$

holds for all columns y such that $u'_i y$ is still a column. We apply it first with $y = x$. Because of condition (2) of the lemma the words $w_{j-1}x$ and u_j fail to satisfy $w_{j-1}x \trianglelefteq u_j$, since p is the least integer such that $w_{j-1}x$ and u_j disagree on the letter on position p. Following Lemma 13 we have $w_{j-1} x u_j \equiv w_{j-1} z u'_j$, where $\mathrm{H}(z) < \mathrm{H}(x)$. Then we have

$$\begin{aligned} &u'_i x u_{i+1} \cdots u_{j-1} u_j \equiv w_i w_{i+1} \cdots w_{j-1} x u_j \\ &\equiv w_i w_{i+1} \cdots w_{j-1} z u'_j \equiv u'_i z u_{i+1} \cdots u_{j-1} u_j u'_j \end{aligned}$$

Because $u'_i z <_{lex} u'_i x$ we obtain a lexicographically smaller representative.

The second possibility is when $|u_i| = p$, $u_{i,\ell} \leq u_{j,\ell}$ for all $1 \leq \ell \leq p$ and $u_{j,p+1} < u_{i,p}$. We still have condition (3) with $u'_i = u_i$. Define $u_j = u'_j u''_j$, where u'_j is the prefix of u_j of length p, and observe that by Lemma 13 the condition $w_{j-1} \trianglelefteq u'_j$ holds. This, via Proposition 8 implies $w_{j-1} u_j \equiv w_{j-1} u''_j u'_j$. Therefore

$$\begin{aligned} &u_i u_{i+1} \cdots u_{j-1} u_j \equiv w_i w_{i+1} \cdots w_{j-1} u_j \\ &\equiv w_i w_{i+1} \cdots w_{j-1} u''_j u'_j \equiv u_i u''_j u_{i+1} \cdots u_{j-1} u'_j \end{aligned}$$

Again $u_i u''_j <_{lex} u_i u_{i+1}$ and we obtain a lexicographically smaller representative.

In order to prove the sufficiency of the property, we need to define the *packing* operation which associates with every P-sequence a unique equivalent contretableau. We consider a P-sequence represented as a sequence of columns $u_1 \trianglelefteq u_2 \trianglelefteq \ldots \trianglelefteq u_n$. The reader is encouraged to have the example below in mind. The contretableau is obtained by pushing all elements of the columns to the right, along the same row, in order to leave no hole between consecutive elements, in other words to right justify all rows. Then the sufficiency will follow from the fact that every sequence can be packed into an equivalent contretableau and that this correspondence is injective.

$$\begin{array}{ccccc} 4 & 5 & 5 & 7 & 8 \\ 3 & & & 5 & 6 \\ 2 & & & & 5 \\ 1 & & & & \end{array} \qquad\qquad \begin{array}{cccccc} & 4 & 5 & 5 & 7 & 8 \\ & & \rightarrow & 3 & 5 & 6 \\ & & \longrightarrow & & 2 & 5 \\ & & \longrightarrow & & & 1 \end{array}$$

Fig. 2. A P-sequence and its equivalent contretableau obtained by packing

We first prove that packing yields a contretableau. From the definition of the P-sequence, all rows are nondecreasing and their length is not increasing from top to bottom. Now, denote by $v_1, v_2, \ldots, v_n$ the n sequences obtained by packing. Fix one of them, say v_i, and consider two entries, $v_{i,\ell}$ and $v_{i,k}$ with $\ell < k$. Then there exists $\alpha \leq \beta \leq i$ such that

$$v_{i,\ell} = u_{\beta,\ell}, v_{i,k} = u_{\alpha,k}$$

Now we have $u_{\beta,\ell} \geq u_{\alpha,\ell} > u_{\alpha,k}$ and therefore $v_{i,\ell} > v_{i,k}$, which proves that the v_i's are columns and the resulting diagram a contretableau.

Given a contretableau, there is a unique way to "unpack" it. Indeed, we construct the rows of the P-sequence one at a time from top to bottom. Assume the r first rows are processed, and consider the $r+1$-th row. Then the leftmost element of the contretableau, say a, can only go under the leftmost element of the P-sequence under construction, which is greater than a. Such an element exists because in the contretableau a is below a greater element. The second leftmost element of the contretableau, say b, goes below the leftmost element of the P-sequence greater than b, and so forth.

$$\begin{array}{cccc} 2&3&4&4\\ 1&2&&\\ &1&& \end{array} \equiv \begin{array}{cccc} 2&3&4&4\\ 1&&2&\\ &&1& \end{array} \equiv \begin{array}{cccc} 2&3&4&4\\ 1&&&2\\ &&&1 \end{array} \equiv \begin{array}{cccc} 2&3&4&4\\ &1&&2\\ &&&1 \end{array} \equiv \begin{array}{cccc} 2&3&4&4\\ &&1&2\\ &&&1 \end{array}$$

Fig. 3. Packing a P-sequence into a congruent contretableau

It remains to prove that the P-sequence and the contretableau are congruent. However, this is obtained as a repetitive application of Proposition 8, and the proof is completed. □

Corollary 14. *The minimal representative of a class has the same column length distribution as its Young tableau.*

Proof. Indeed, Young tableaux and contretableaux have the same column length distribution by Greene's invariant Theorem, cf. [9] (this theorem asserts that the sequence, over k, of the maximum sums of lengths of k disjoint columns, is an invariant of the congruence class). The above construction shows that the minimal representative has the same column distribution as its contretableau. □

3.2 Complexity issues

Here we consider the effective construction of the minimal representative as a consequence of Theorem 12.

A naive method to obtain the minimal representative in the class of a Young tableau would be to apply the inverse operation of inserting an element in a tableau. More precisely, start from the element in one of the corners of a Young tableau. In Example 6 there are 3 corners all labeled by 5. From one chosen corner on, process the columns from right to left. Substitute the element, say a, on the corner for the element of the column to its left which is the highest element less than or equal to a. If b is this element repeat the process with b instead of a, and next column. This results in pushing elements from column to column, and expelling an element to the left of the Young tableau. This element is a possible first letter of a word in the equivalence class.

	5	5
3 4 5	4 5	4 5 5
(5) 1 2 4 5	(3) 1 2 4 5	(3) 1 2 4

Fig. 4. Extracting a possible first letter from the Young tableau of Example 6

In Example 6, starting from the three corners, from top to bottom, yields 3 possible first letters, namely 5, 3 and 3. The first decomposition is ruled out. At this point we know for sure that the minimal representative starts with the letter 3. However, canceling this letter leads to two nonequivalent Young tableaux and it is not clear whether to compute the next letter from the second or from the third decomposition.

Proposition 15. *Given a word w of length n, there exists an $\mathcal{O}(n^{\frac{3}{2}})$ algorithm which finds the lexicographically minimal representative equivalent to w.*

Proof. Given a word w, construct its contretableau. If n is the length of w, its construction as sketched in paragraph 2.3 has complexity $\mathcal{O}(n^{\frac{3}{2}})$. The unpacking operation is similar to a merge of k arrays, where each array is a row of the contretableau. This costs again $\mathcal{O}(n)$ operations. □

3.3 Application to the cross-section

Consequently we get the main result.

Theorem 16. *The set of alphabetically minimal words of the plactic congruence over an arbitrary finite alphabet is regular.*

Proof. An informal description of the automaton will do. The set of columns define a (suffix) code. Read the word and perform its decomposition into maximal columns. Record all different columns encountered in the order of appearance. If a next column, say u is not the last column recorded, or if it fails to satisfy the condition $v \trianglelefteq u$ with all columns v recorded, stop. Otherwise, put u in the record. If the word can be read off entirely, it is minimal. □

4 Final remarks

The purpose of this last section is twofold. The fact that the lexicographic cross-section is regular, is remarkable to the extent that most possible related constructions cannot be recognized by finite memory machines. The second remark concerns an interesting property of yet another classical relation and may be viewed as an invitation to investigate the field further.

4.1 Natural binary relations on the plactic monoid

None of the following natural binary relations is rational, in the sense that there is no two-tape finite automata recognizing them.

EQUIVALENCE $\{(x, y) \in \Sigma^* \times \Sigma^* \mid x \equiv y\}$

MINIMIZATION $\{(x, y) \in \Sigma^* \times \Sigma^* \mid x \equiv y$ and y is lexicographically minimal$\}$,

MULTIPLICATION for some $a \in \Sigma$,

$$\{(x, y) \in \Sigma^* \times \Sigma^* \mid xa \equiv y \text{ and } x, y \text{ are lexicographically minimal}\}$$

The proof of these claims is a simple exercise if one has in mind Eilenberg's result on length preserving relations that are recognized by two-tape automata, see [7, Thm IX. 6.3].

4.2 The relation of conjugacy

We recall that two elements x, y of a monoid are *conjugate*, if there exists an element z such that $xz = yz$ holds, written as $C(x, y)$, and that they are *transposed*, if there exist u, v such that $x = uv$ and $y = vu$, written as $T(x, y)$. It is a simple exercise to verify that the former relation is reflexive and transitive, while the latter is reflexive and symmetric. For free monoids, these relations coincide, see [14]. This is no longer the case for general monoids. The only claim that can be made in all generality is that the transitive closure of the relation of transposition is always included in the conjugacy relation. In the case of the trace monoids equality $C = T^k$ holds where k is the diameter of the graph of noncommutations, [6]. In the present case it is still true that the relation of conjugacy is the transitive closure of the relation of transposition, but we are able to bound the number of compositions by a parameter depending on the size of the alphabet. Consequently, the conjugacy relation is an equivalence relation.

Theorem 17. *The equality* $C = T^{2(k-1)}$ *holds where* $k = |\Sigma|$.

Proof. We prove a stronger result which implies the theorem. Let p be the number of different symbols of an element x of the plactic monoid. We show that $(x, y) \in \mathrm{T}^p$ where y is the commutative image of x, i.e., $y = 1^{|x|_1} 2^{|x|_2} \cdots k^{|x|_k}$.

Indeed, let $1 \leq a_1 < \ldots < a_p \leq k$ be the ordered set of all the different letters occurring in x. Let $\lambda_1 \lambda_2 \cdots \lambda_r$ be the concatenation of the rows of its

Young tableau from top to bottom. Clearly we may assume that $r > 1$ since otherwise we are done. All $|x|_1$ occurrences of a_1 are the first letters of row λ_r. Consider the word $w = \lambda_r\lambda_1\lambda_2 \cdots \lambda_{r-1}$ which is a representative of a conjugate. By Schensted's rules, all $|x|_1$ occurrences of a_1 are followed by all $|x|_2$ occurrences of a_2, and these are the first $|x|_1 + |x|_2$ letters of the bottom row of the Young tableau associated with w. It suffices to carry on this process of putting the last row to the left of the remaining rows at most $p-1$ times in order to get the result. □

References

1. Anatoly V. Anisimov and Donald E. Knuth. Inhomogeneous sorting. *International Journal of Computer and Information Sciences*, 8(4):255–260, 1979.
2. André Arnold, Mathias Kanta, and Daniel Krob. Recognizable subsets of the two letter plactic monoid. *Information Processing Letters*, 64:53–59, 1997.
3. Pierre Cartier and Dominique Foata. Problèmes combinatoires de commutation et réarrangements. *Lecture Notes in Mathematics*, 85, 1969.
4. Christian Choffrut and Robert Mercaş. Contextual partial commutations. *Discrete Mathematics and Theoretical Computer Science*, 12(4):59–72, 2010.
5. Volker Diekert and Grzegorz Rozenberg. *The Book of Traces*. World Scientific Publishing Co., Singapore, 1995.
6. Christine Duboc. On some equations in free partially commutative monoids. *Theoretical Computer Science*, 46(2–3):159–174, 1986.
7. Samuel Eilenberg. *Automata, Languages and Machines*, volume A. Academic Press, 1974.
8. David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston. *Word processing in groups*. Jones and Bartlett, 1992.
9. Curtis Greene. An extension of Schensted's Theorem. *Advances in Mathematics*, 14(2):254 – 265, 1974.
10. Donald E. Knuth. *The art of computer programming*, volume 3. Addison Wesley, 1973.
11. Gérard Lallement. *Semigroups and Combinatorial Applications*. John Wiley & Sons, 1979.
12. Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon. The plactic monoid. In M. Lothaire, editor, *Algebraic Combinatorics on Words*, pages 144–173. Cambridge University Press, 2002.
13. Alain Lascoux and Marcel-Paul Schützenberger. Le monoïde plaxique. In Aldo de Luca, editor, *Non-commutative Structures in Algebra and Geometric Combinatorics*, volume 109, pages 129–156. C.N.R., 1981.
14. André Lentin and Marcel-Paul Schützenberger. A combinatorial problem in the theory of free monoids. In R. C. Bose and T. E. Bowlings, editors, *Combinatorial Mathematics*, pages 112–144. North Carolina Press, Chapel Hill, N. C., 1967.