Reducing the Ambiguity of Parikh Matrices ⋆

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Abstract

The Parikh matrix mapping allows us to describe words using matrices. Whilst compact, this description comes with a level of ambiguity since a single matrix may describe multiple words. In this paper, we investigate how considering the Parikh matrices of various transformations of a given word can decrease that ambiguity. More specifically, for any word, we study the Parikh matrix of its projection to a smaller alphabet as well as that of its Lyndon conjugate. Our results demonstrate that ambiguity can often be reduced using these concepts, and we give conditions on when they succeed.

1. Introduction

Parikh vectors [1] are a well established and thoroughly investigated way to represent the number of occurrences of letters in a sequence of symbols (or word). While they are easily computed and guaranteed to be logarithmic in the size of the word they represent, they are inevitably ambiguous; that is, as soon as words consist of at least two different letters, multiple words share the same Parikh vector.

Parikh matrices [2] are an extension of Parikh vectors that reduce this ambiguity. To this end, they do not only contain the frequency of the individual letters in the word, but they also include the numbers of some of its (scattered) subwords, namely of all those subwords which consist of a consecutive sequence of distinct letters that follow some fixed order on the underlying alphabet. For example, if we consider a word over the alphabet $\{a, b, c\}$ and the common lexicographical order on this alphabet, then its Parikh matrix contains the frequencies of a, b and c (which corresponds

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to the information held by the word's Parikh vector) as well as the numbers of occurrences of the subwords ab, bc and abc. Such a matrix still has the same asymptotic compactness as a Parikh vector, but it is associated to a significantly smaller number of words. For example, while the words $v = abba$ and $w = aabb$ share a common Parikh vector, they have different Parikh matrices, as v contains two occurrences of the subword ab, whereas w has four of them. However, the additional information included in Parikh matrices does not normally remove their ambiguity entirely: for instance, the word $v' = baab$ has the same number of occurrences of a, b and ab as the word v introduced above, and therefore shares a common Parikh matrix with v.

A substantial part of the literature on the Parikh matrix mapping investigates this remaining ambiguity. A major focus of the related research is the study of refined versions of Parikh matrices that would render the mapping from a word less ambiguous or potentially even injective [3–9], for example through the use of polynomials or various extensions of the mapping. Due to the significant combinatorial complexity of the subject, most such attempts (as well as findings on basic Parikh matrices) are specific to words over binary [10–14] or ternary [15–18] alphabets, leaving alphabets of size greater than three relatively unexplored.

In terms of reducing the ambiguity of the Parikh matrix of a given word, the authors of [2, 12] gather more information about the specific word by altering the order of the alphabet, known as the dual order. In [12] they also consider the Parikh matrix of the reverse image of the word. Hence, these approaches either modify the alphabet or the word itself, and they study to which extent the ambiguity of the resulting Parikh matrix can be reduced through these changes. In the present paper, we pursue a similar avenue, but we alter the word more radically. Our approach yields two new concepts, namely the $\mathbb{P}$-Parikh matrices and $\mathbb{L}$-Parikh matrices, and we demonstrate that they effectively reduce the ambiguity of the Parikh matrix mapping in most cases.

In the $\mathbb{P}$-Parikh matrix mapping, we apply a projection morphism to the given word, i. e., we delete selected letters from it and consider the Parikh matrix of the resulting word over the reduced alphabet. This represents a particular case of the extended mapping introduced in [9]. For example, if we consider the words $abcaabaac$ and $abacabcaa$, then it can be easily verified that they share the same number of letters and subwords ab, bc and abc, rendering their Parikh matrices identical and therefore ambiguous. However, the $\mathbb{P}$-Parikh matrices associated to them with respect to the alphabet $\{a, c\}$ include the number of subwords ac, which is 6 in the former, but only 5 in the latter of the words. Hence, there exist $\mathbb{P}$-Parikh matrices not shared by the words.

We show that, using $\mathbb{P}$-Parikh matrices, we can reduce the ambiguity of the vast majority of words. We also explore when $\mathbb{P}$-Parikh matrices are ineffective, and we provide insights into the types of words that cannot be uniquely described by a $\mathbb{P}$-Parikh matrix.

Since $\mathbb{P}$-Parikh matrices are defined for a subset of the initial alphabet, they prove inadequate when dealing with binary alphabets. We therefore consider an alternative transformation of words: the Lyndon conjugate, first introduced in [19], which is defined as the lexicographically smallest circular rotation of a word. Lyndon conjugates have previously been used as a tool for ambiguity reduction. In [14], the authors define the Lyndon image of a Parikh *matrix* as the lexicographically smallest word describing such a matrix. Hence every Parikh matrix has exactly one distinct Lyndon image, which therefore allows each Parikh matrix to be described uniquely. In the context of this paper, we use the Lyndon conjugate differently, i. e., we consider the Parikh matrix of the Lyndon conjugate of a *word*, and we call the resulting matrix the $\mathbb{L}$-Parikh matrix of the original word.

If we consider the Parikh matrix of the Lyndon conjugates of the two previously given words, then we can observe that $aabaacabc$ has 7 occurrences of ab, whereas $aaabacabc$ has 8, making their Parikh matrices different. Hence, the ambiguity of their Parikh matrix can be reduced using $\mathbb{L}$-Parikh matrices.

While $\mathbb{L}$-Parikh matrices are a useful concept for any alphabet size, we focus on the cases where they reduce ambiguity in the binary alphabet and show that this happens for most words over such alphabets. We give specific conditions of when $\mathbb{L}$-Parikh matrices do not help to reduce the ambiguity of the given word, and investigate the words for which these criteria apply. This leads us to our main result of the paper, a characterisation of words whose ambiguity can be reduced using $\mathbb{L}$-Parikh matrices.

Our paper is organised as follows: In Section 2 we present some basic definitions and notions, including some basic properties of Parikh matrices. Section 3 examines the first of the two notions we introduce, the $\mathbb{P}$-Parikh matrix, establishing conditions for when it can or cannot achieve ambiguity reduction. In Section 4, we study equivalent questions for $\mathbb{L}$-Parikh matrices, largely focusing on binary alphabets where appropriate. We end our paper with conclusions as well as directions for future work.

2. Preliminaries

It is assumed the reader is familiar with the basics of combinatorics on words. If needed, [20] can be consulted. Throughout this paper, $\mathbb{N}$ refers to the set of natural numbers starting with 1.

We refer to a string of arbitrary letters as a *word* which is formed by the concatenation of letters. The set of all letters used to create our words is called an *alphabet*. We represent an *ordered alphabet* as $\Sigma_k = \{a_1 < a_2 < \cdots < a_k\}$, where $k \in \mathbb{N}$ is the *size* of the alphabet, and by convention a_i is the ith letter in the Latin alphabet. Whenever the alphabet size is irrelevant or understood, we omit this from notation, using only Σ. All alphabets referred to in this paper have an order imposed on them. The *Kleene star*, denoted *, is the operation that, once applied to a given alphabet, generates the set of all finite words that result from concatenating any letters from that alphabet.

We denote the concatenation of two words u and v as uv. The *length* of a word is the total number of, not necessarily distinct, letters it contains and the *empty word*, of length zero, is referred to as ε. If we wish to consider the i^{th} letter in a word w, then we use the notation $w[i]$.

The *reversal* of a word $w = w[1]w[2]\cdots w[m]$, where $w[i] \in \Sigma$ for all i, $1 \leq i \leq m$, is denoted $rev(w)$ and defined as $rev(w) = w[m]w[m-1]\cdots w[1]$. We say that v is a *factor* of w if and only if w can be written as $w = w_1 v w_2$ for some $w_1, w_2 \in \Sigma^*$. If $w_1 = \varepsilon$, then we also call v a *prefix* of w, and if $w_2 = \varepsilon$, then v is a *suffix* of w. A word $u = u[1]u[2]\cdots u[m]$, where $u[1], u[2] \ldots, u[m] \in \Sigma$, is a *subword* of a word v if there exists factors $v_0, v_1, \ldots, v_m \in \Sigma^*$ such that $v = v_0 u[1] v_1 u[2] \cdots v_{m-1} u[m] v_m$. We use $|v|_u$ to denote the number of distinct occurrences of u as a subword in v. We say that a word $u \in \Sigma^*$ is *lexicographically smaller* than a word $v \in \Sigma^*$, denoted $u <_{lex} v$, if $u \neq v$ and either u is a prefix of v or, for the smallest i satisfying $u[i] \neq v[i]$, the letter $u[i]$ precedes the letter $v[i]$ in the order on Σ.

The *Parikh vector* [1] associated with a word w is obtained through a mapping $\phi : \Sigma^* \to \mathbb{N}^k$, defined as $\phi(w) = [|w|_{a_1}, |w|_{a_2}, \ldots, |w|_{a_k}]$. For a matrix M of size $k \times k$, the *j-diagonal* is defined as all elements of M that are in the position $M_{i,i+j}$, for $i = 1, 2, \ldots, k-j$.

We now give the definition of a Parikh matrix, followed by the method by which a Parikh matrix may also be calculated.

Definition 1 ([2]). *Let M_{k+1} denote the set of all square matrices of size $(k+1) \times (k+1)$, where k is the size of the ordered alphabet $\Sigma = a_1 < a_2 < \cdots < a_k$. The* Parikh matrix mapping *is the morphism $\Psi : \Sigma^* \to M_{k+1}$, defined as follows. For $a_q \in \Sigma$ with q representing where in the* *ordered alphabet the letter lies, if* $\Psi(a_q) = (m_{i,j})_{1\leq i,j\leq k+1}$*, then for each* $1 \leq i \leq k+1$*,* $m_{i,i} = 1$*,* $m_{q,q+1} = 1$*, and all other elements of the matrix* $\Psi(a_q)$ *are zero.*

A word is *associated* with a matrix, called its Parikh matrix, if the matrix is obtained from that word following the process. Let $w \in \Sigma_k^*$. The *Parikh matrix*, denoted $\Psi(w)$, that w is associated with has size $(k+1) \times (k+1)$. The diagonal of the matrix is populated with 1's and all elements below it are 0. The count of all subwords that consist of consecutive letters in Σ_k and are of length n in the word are found, lexicographically ordered, on the n-diagonal, for $1 \leq n \leq k$. Observe that when a Parikh matrix is calculated in this way, the Parikh vector is found on the 1-diagonal of the corresponding matrix. This concept is encapsulated in the following Theorem.

Theorem 2 ([21])**.** *Let* $\Sigma = \{a_1 < a_2 < \cdots < a_k\}$ *be an ordered alphabet, where* $k \geq 1$*, and assume that* $w \in \Sigma^*$*. The matrix* $\Psi(w)$ *has the following properties:*

1. $m_{i,j} = 0$*, for all* $1 \leq j < i \leq (k+1)$*;*
2. $m_{i,i} = 1$*, for all* $1 \leq i \leq (k+1)$*;*
3. $m_{i,j} = |w|_u$ *where* $u = a_i a_{i+1} \dots a_{j-2} a_{j-1}$*, for all* $1 \leq i < j \leq (k+1)$.

We shall illustrate Definition 1 in Example 3 below.

One notion we introduce in this paper relies on a change in alphabet. As such, to emphasise the alphabet Σ used for obtaining a Parikh matrix, we write $\Psi_\Sigma(w)$. If no confusion arises, we shall omit the alphabet from the notation, in favour of legibility, and write $\Psi(w)$.

We say that a Parikh matrix *describes* a word if the word is associated to the matrix. Notice that due to the associativity of matrix multiplication, the Parikh matrix of a word can be constructed from the Parikh matrices of its factors. For a word $w = u_1u_2$, we have $\Psi_{\Sigma_n}(w) = \Psi_{\Sigma_n}(u_1)\Psi_{\Sigma_n}(u_2)$.

Example 3. *Consider the word* $w = abcda$ *defined over the alphabet* $\Sigma_4 = \{a < b < c < d\}$*. Then by definition our Parikh matrix is of size* 5×5 *and we have*

$$\begin{aligned}\Psi(abcda) &= \Psi(a) \cdot \Psi(b) \cdot \Psi(c) \cdot \Psi(d) \cdot \Psi(a)\\ &= \begin{pmatrix} 1&1&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0&0\\ 0&1&1&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&1&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&1\\ 0&0&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 1&1&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix} = \begin{pmatrix} 1&2&1&1&1\\ 0&1&1&1&1\\ 0&0&1&1&1\\ 0&0&0&1&1\\ 0&0&0&0&1 \end{pmatrix} \quad \triangleleft\end{aligned}$$

For the rest of this work we refine our notation for a Parikh matrix where we remove the elements not depending on the associated word. By definition, a Parikh matrix is an upper triangular matrix with 1's on the diagonal and zeroes underneath, regardless of the word described. For aesthetics, removing the redundant part leaves us with a triangular structure that holds the same information as the original matrix. Thus, we shall represent the Parikh matrix from Example 3 as

$$\Psi(abcda) = \left\langle \begin{smallmatrix} 2 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{smallmatrix} \right\rangle .$$

We say that two words w and w' are *conjugates* if we can write $w = uv$ and $w' = vu$. For a word w, the *conjugacy class of* w, denoted $C(w)$, is the class of all of its possible conjugates. A *conjugacy class is associated to a certain Parikh matrix* if at least one word belonging to that class is associated to the matrix. We observe that it is possible that two or more conjugacy classes are associated to a single Parikh matrix.

Example 4. *The matrix from Example 3,* $\left\langle \begin{smallmatrix} 2 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{smallmatrix} \right\rangle$*, has only the words abacd, abcad, and abcda associated to it. Each of these words are members of different conjugacy classes. Hence this matrix has three conjugacy classes associated to it.* ◁

A Parikh matrix can describe multiple words, as seen above, although cases exist where a matrix describes a single word, e. g., *aabb* is the only word associated to $\langle \begin{smallmatrix} 2 & 4 \\ & 2 \end{smallmatrix} \rangle$. We say that two words are *amiable* if they are associated to the same Parikh matrix. If two or more words are associated to a single Parikh matrix, we say that the matrix is ambiguous. Later in this paper, we reduce the ambiguity of a word using both its Parikh matrix and the Parikh matrix of an altered form of that word to describe it. Therefore, we introduce a formal definition of the *ambiguity* that multiple functions may have based on the set of all words that satisfy all functions.

Definition 5. *We define* $\mathcal{A}(w, f_1, f_2, \ldots, f_n) = \{v | f_i(v) = f_i(w) \text{ for all } 1 \leq i \leq n\}$ *for a word* w *and functions* $f_1, f_2, \ldots, f_n$*. If* $|\mathcal{A}(w, f_1, f_2, \ldots, f_n)| = 1$*, then we call* w unambiguous *on* $f_1, f_2, \ldots, f_n$*, and say that* $f_1(w), f_2(w), \ldots, f_n(w)$ uniquely define w*. However, if we have that* $|\mathcal{A}(w, f_1, f_2, \ldots, f_n)| > |\mathcal{A}(w, f_1, f_2, \ldots, f_m)|$ *for* $m > n$ *and functions* $f_{n+1}, f_{n+2}, \ldots, f_m$*, then we say that* $f_{n+1}, f_{n+2}, \ldots, f_m$ reduce the ambiguity of w *on* $f_1, f_2, \ldots, f_n$*.*

Observe that whenever $m > n$, we always have $|\mathcal{A}(w, f_1, f_2, \ldots, f_n)| \geq |\mathcal{A}(w, f_1, f_2, \ldots, f_m)|$. Furthermore, if $|\mathcal{A}(w, f_1, f_2, \ldots, f_n)| = 1$, then $\mathcal{A}(w, f_1, f_2, \ldots, f_n)$ is unambiguous and thus it is not possible for $f_{n+1}, f_{n+2}, \ldots, f_m$ to reduce ambiguity any further.

First we introduce the $\mathbb{P}$-Parikh matrix. The $\mathbb{P}$-Parikh matrix is in essence the Parikh matrix of a projection of a word, and represents a particular case of the extension of the Parikh matrix mapping presented in [18]. For $n \in \mathbb{N}$, $w \in \Sigma_n^*$ and $S \subset \Sigma_n$, the $\mathbb{P}$-Parikh matrix of w with respect to S is defined as follows.

Definition 6. *For $m, n \in \mathbb{N}$ with $1 \leq m \leq n$, let $S \subset \Sigma_n$ such that $S = \{a_{k_1}, a_{k_2}, \ldots, a_{k_m}\}$, where $0 < k_1 < k_2 < \cdots < k_m \leq n$. We define the $\mathbb{P}$-Parikh matrix of the word w with respect to S as $\Psi_S(\pi_S(w))$, where the morphism $\pi : \Sigma_n^* \to \Sigma_m^*$ is defined as*

$$\pi_S(a_i) := \begin{cases} a_i & : a_i \in S, \\ \varepsilon & : a_i \notin S. \end{cases}$$

To gain some intuition about the above definition, we consider an example.

Example 7. *As in Example 3, let $\Sigma_4 = \{a, b, c, d\}$ and $w = abcda$; furthermore, let $S = \{a, c\}$. We obtain the $\mathbb{P}$-Parikh matrix $\Psi_S(w)$ by determining the Parikh matrix of $\pi_S(w)$:*

$$\pi_S(w) = \pi_S(a) \cdot \pi_S(b) \cdot \pi_S(c) \cdot \pi_S(d) \cdot \pi_S(a) = a\varepsilon c\varepsilon a = aca$$

$$\Psi_{\{a,c\}}(\pi_{\{a,c\}}(abcda)) = \Psi_{\{a,c\}}(aca) = \left\langle \begin{smallmatrix} 2 & 1 \\ & 1 \end{smallmatrix} \right\rangle. \qquad \triangleleft$$

The *Lyndon conjugate* of a word w, denoted $L(w)$, is the conjugate that is lexicographically smallest based on the order on the alphabet. Next, we introduce the $\mathbb{L}$-Parikh matrix associated to a word.

Definition 8. *Given a word w, we define its* $\mathbb{L}$-Parikh matrix, *Ψ_L, as the Parikh matrix associated with its Lyndon conjugate, $L(w)$.*

In order to illustrate this definition, we again consider the word from Example 3:

Example 9. *Let $\Sigma_4 = \{a, b, c, d\}$ and $w = abcda$. Then $L(w) = aabcd$, and we calculate the $\mathbb{L}$-Parikh matrix of w as the Parikh matrix of $L(w)$,*

$$\Psi_L(abcda) = \Psi(aabcd) = \left\langle \begin{smallmatrix} 2 & 2 & 2 & 2 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{smallmatrix} \right\rangle.$$

It is shown in [22] (see [10], for an explicit description) that there exist transformations that, when applied to a word, create a new word that is amiable with the original. For non-binary alphabets, a Type 1 transformation is given.

Lemma 10 ([22])**.** *Let $w, w' \in \Sigma_n^*$ with $n \geq 3$. Then w transforms into w' using a* Type 1 *transformation if $w = u_1 a_i a_j u_2$ and $w' = u_1 a_j a_i u_2$, where $u_1, u_2 \in \Sigma_n^*$, $a_i, a_j \in \Sigma_n$, and $|i - j| \geq 2$.*

For binary alphabets, in [10], a second type of transformation is described, referred to as a Type 2, that allows us to check if certain words are amiable without constructing their matrices.

Lemma 11 ([10]). *Let $w, w' \in \Sigma_2^*$. Then w transforms into w' through a* Type 2 *transformation if $w = xa_1a_2ya_2a_1z$ and $w' = xa_2a_1ya_1a_2z$, or vice-versa, where $x, y, z \in \Sigma_2^*$, $a_1 \neq a_2$ and $a_1, a_2 \in \Sigma_2$.*

It was proven that Type 2 transformations characterise all binary amiable words, but not all ternary words.

Remark 12. *Since binary amiable words are fully characterised by* Type 2 *transformations, that is we can obtain one amiable word from another through a series of* Type 2 *transformations, for a binary word $w \in \Sigma_2^*$, if w is unambiguous, namely $|\mathcal{A}(w, \Psi)| > 1$, then $|w|_a, |w|_b \geq 2$, since otherwise such transformations cannot be applied.*

For ternary alphabets, the authors give in [15] an extension of Lemma 11. We make use of this theorem in the proofs presented later on.

Theorem 13 ([15]). *Let $w \in \Sigma_3^+$ be a nonempty word and t an integer with $t \geq 1$ such that w contains t occurrences of factors of the form $\beta_k = a_{i_k}a_{j_k}\alpha_k a_{j_k}a_{i_k}$, for $1 \leq k \leq t$, where*

- *$\alpha_k \in \Sigma_3^*$, a_{i_k},$a_{j_k} \in \Sigma_3$, $|i_k - j_k| = 1$ $(1 \leq k \leq t)$;*
- *If $w = x\beta_n y = x'\beta_r y'$ with $x, y, x', y' \in \Sigma_3^*$ and $n \neq r$, then $||x| - |x'|| > 1, ||x\alpha_n| - |x'\alpha_r|| > 1$;*
- *$\sum_{k=1}^{t}(j_k - i_k)(j_k + i_k - 4)|\alpha_k|_{a_{6-i_k-j_k}} = 0$.*

Then w and w' are amiable, where w transforms into w' through a series of Type 2 *transformations by replacing all occurrences of β_k with $\beta'_k = a_{j_k}a_{i_k}\alpha_k a_{i_k}a_{j_k}$.*

3. $\mathbb{P}$-Parikh Matrices

In this section, we examine when and by how much $\mathbb{P}$-Parikh matrices help reduce the ambiguity of a given word. When we refer to a reduction in ambiguity using $\mathbb{P}$-Parikh matrices, we mean that the number of words described by the original Parikh matrix and their respective $\mathbb{P}$-Parikh matrices for some set $S \subset \Sigma_n$, is strictly less than the total number of words described by the original Parikh matrix alone, i. e., $|\mathcal{A}\{w, \Psi_{\Sigma_n}, \Psi_S\}| < |\mathcal{A}\{w, \Psi_{\Sigma_n}\}|$.

First we present an example of a $\mathbb{P}$-Parikh matrix removing the ambiguity of a Parikh matrix entirely.

Example 14. *Consider the word $w_1 = abcda$ from Example 3, which is amiable with the words $w_2 = abcad$, $w_3 = abacd$ and no others. Then we choose our set $S = \{a, c\}$ and obtain*

$$\Psi_S(\pi_S(w_1)) = \Psi_S(aca) = \langle \begin{smallmatrix} 2 & 1 \\ & 1 \end{smallmatrix} \rangle$$
$$\Psi_S(\pi_S(w_2)) = \Psi_S(aca) = \langle \begin{smallmatrix} 2 & 1 \\ & 1 \end{smallmatrix} \rangle$$
$$\Psi_S(\pi_S(w_3)) = \Psi_S(aac) = \langle \begin{smallmatrix} 2 & 2 \\ & 1 \end{smallmatrix} \rangle .$$

Therefore w_3 has a different Parikh matrix than w_1 and w_2, and we can in fact uniquely describe w_3 using a combination of its Parikh matrix and $\mathbb{P}$-Parikh matrix when $S = \{a, c\}$. ◁

We next introduce some terms that are useful when describing how effective a given $\mathbb{P}$-Parikh matrix is at reducing ambiguity.

Definition 15. *Given a word $w \in \Sigma_n^*$, we call $\Psi(w)$* $\mathbb{P}$-distinguishable *if either $|\mathcal{A}(w, \Psi)| = 1$ or there exists a word $u \in \Sigma_n^*$ and a set $S \subset \Sigma_n$ such that $\Psi_\Sigma(w) = \Psi_\Sigma(u)$ and $\Psi_S(w) \neq \Psi_S(u)$. In the latter case we say that w and u are* $\mathbb{P}$-distinct*, with respect to S. Furthermore, we call w* $\mathbb{P}$-unique *if there exist sets $S_1, S_2, \ldots, S_m \subset \Sigma_n$ such that $|\mathcal{A}(w, \Psi, \Psi_{S_1}, \Psi_{S_2}, \ldots, \Psi_{S_m})| = 1$.*

Now we use these terms to examine words whose ambiguity can be reduced using $\mathbb{P}$-parikh matrices, namely those that contain any length two factor where the two letters they consist of are not equal or consecutive in the alphabet.

Proposition 16. *For any word $w \in \Sigma_n^*$ with a factor $a_i a_j$ where $|i - j| > 1$, we have that $\Psi(w)$ is $\mathbb{P}$-distinguishable.*

Proof. Since $|i - j| > 1$, if $w = u_1 a_i a_j u_2$ where $u_1, u_2 \in \Sigma_n^*$, then $w' = u_1 a_j a_i u_2$ is amiable to w, following Lemma 10. Without loss of generality, take $S = \{a_i < a_j\}$. Then $\Psi_S(\pi_S(w)) \neq \Psi_S(\pi_S(w'))$, since $|w|_{a_i a_j} = |\pi_S(w)|_{a_i a_j}$, $|w|_{a_i a_j} \in \Psi_S(\pi_S(w))$ and $|w'|_{a_i a_j} \in \Psi_S(\pi_S(w'))$, and $|w|_{a_i a_j} \neq |w'|_{a_i a_j}$. □

It is simple to identify words that have such factors by comparing adjacent positions in the word. We can use this to find a lower bound for the proportion of words that are uniquely identified for a given alphabet and word length.

Proposition 17. *The number of words of length m in Σ_n that are reduced in ambiguity by $\mathbb{P}$-Parikh matrices is bounded below by $(n^m) - (n \times 3^{m-1})$.*

Proof. The result is straightforward. To see that this is true, note that in a word for which the ambiguity cannot be reduced by $\mathbb{P}$-Parikh matrices except for the first letter all of the other $m-1$ positions are depending on the previous one, since they can be only equal to it, or the letters immediately preceding or following it in the lexicographical order, following Proposition 16, i.e., if a_i occurs in position j, then the $j+1$th letter is in the set $\{a_i, a_{i-1}, a_{i+1}\}$. So, the stated bound for the number of $\mathbb{P}$-distinguishable words follows. □

Notice especially that as the alphabet size and word length get larger, the proportion of words which are reduced in ambiguity by $\mathbb{P}$-Parikh matrices also gets larger. We therefore conclude that the use of $\mathbb{P}$-Parikh matrices reduces ambiguity for a larger ratio of words for bigger alphabets rather than smaller.

There also exist words for which $\mathbb{P}$-Parikh matrices do not reduce ambiguity. The following, straightforward observation says that if our choice of a subset consists of only consecutive letters of the initial alphabet, the corresponding $\mathbb{P}$-Parikh matrices are not $\mathbb{P}$-distinguishable.

Remark 18. *If all elements in $S \subset \Sigma_n$ are consecutive in Σ, then $|\mathcal{A}(w, \Psi_{\Sigma_n})| = |\mathcal{A}(w, \Psi_{\Sigma_n}, \Psi_S)|$.*

The result of Remark 18 strengthens the one of Proposition 16 by telling us that the ambiguity of words defined over binary alphabets is not reducible by any $\mathbb{P}$-Parikh matrix.

Corollary 19. *There does not exist a Parikh matrix that describes binary words whose ambiguity can be reduced by $\mathbb{P}$-Parikh matrices.*

Furthermore, there exist non-binary words for which $\mathbb{P}$-Parikh matrices do not remove ambiguity, namely those that are not $\mathbb{P}$-unique. We end this section by giving two classes of words whose ambiguity are not reducible by $\mathbb{P}$-Parikh matrices, no matter how we choose the set S.

Proposition 20. *Consider two words $w, w' \in \Sigma_n^*$ with the form $w = u_1 a_i a_j v a_j a_i u_2$ and $w' = u_1 a_j a_i v a_i a_j u_2$, where $a_i \leq a_j \in \Sigma_n$ and $u_1, u_2 \in \Sigma_n^*$. If $v \in \{a_k \in \Sigma_n | a_i \leq a_k \leq a_j\}^*$, then for all $S \subseteq \Sigma_n$, we have $\Psi_S(\pi_S(w)) = \Psi_S(\pi_S(w'))$.*

Proof. Firstly, if $a_i = a_k = a_j$, equivalence clearly follows, as $w = w'$. Now, let $a_i < a_j$.

In the case where S contains either only a_i or a_j, then $\pi_S(w) = \pi_S(w')$ since a_i and a_j are the only letters that swap places in w' compared to w. Since $\pi_S(w) = \pi_S(w')$, clearly $\Psi_S(\pi_S(w)) = \Psi_S(\pi_S(w'))$ follows.

If $S = \{a_i, a_j\}$, then, $\pi_S(w)$ is a binary word and can be transformed via a Type 2 transformation, from Lemma 11, into $\pi_S(w')$, so $\Psi_S(\pi_S(w)) = \Psi_S(\pi_S(w'))$.

Next consider that $\{a_i, a_j\} \subset S$, $|S| > 2$, and S contains no elements between a_i and a_j. Then $\pi_S(w) = \pi_S(u_1)a_ia_ja_ja_i\pi_S(u_2)$ and $\pi_S(w') = \pi_S(u_1)a_ja_ia_ia_j\pi_S(u_2)$. Using an extension from [15, Theorem 3] of the Type 2 transformations we can transform $\pi_S(w)$ into $\pi_S(w')$, and get that $\Psi_S(\pi_S(w)) = \Psi_S(\pi(w'))$.

Finally, consider the case where S contains a_i, a_j, and at least one letter that comes lexicographically between them. Then, $\pi_S(w)$ can be transformed into $\pi_S(w')$ via two Type 1 transformations on a_i and a_j, since a_i and a_j are not lexicographically adjacent in S (see Lemma 10). □

Before we give our next result, we first introduce an observation below that is used in the proof of Proposition 22.

Remark 21. *Let $\Sigma = \{a_1 < a_2 < \cdots < a_k\}$. Then for all $1 \leq i < k$ the following equation must hold for any word $w \in \Sigma^*$.*

$$|w|_{a_ia_{i+1}} + |w|_{a_{i+1}a_i} = |w|_{a_i}|w|_{a_{i+1}} \tag{1}$$

The ideas presented in Proposition 20 give rise to another class of words whose ambiguity is not reducible by $\mathbb{P}$-Parikh matrices, by loosening the condition on v and extending the length of the word.

Proposition 22. *Consider two words $w, w' \in \Sigma_n^*$ with the form $w = u_1a_ia_jv_1a_ja_ia_ja_iv_2a_ia_ju_2$, and $w' = u_1a_ja_iv_1a_ia_ja_ia_jv_2a_ja_iu_2$, where $a_i < a_j \in \Sigma_n$ and $u_1, u_2, v_1, v_2 \in \Sigma_n^*$. Let $v_1 = v_1[1]v_1[2]\cdots v_1[x]$ and $v_2 = v_2[1]v_2[2]\cdots v_2[y]$. Then, w and w' are not $\mathbb{P}$-distinct if and only if $|v_1|_{a_\ell} = |v_2|_{a_\ell}$ for all $a_\ell \notin \{a_k | a_i \leq a_k \leq a_j\}$, and at least one of the following conditions is true:*

1. *$v_1, v_2 \in \{a_k | a_k \leq a_j\}^*$, and for $\ell < p$, if $v_2[p], v_2[\ell] \in \{a_k | a_k < a_i\}$, then $v_2[p] \leq v_2[\ell]$, and if $v_1[p], v_1[\ell] \in \{a_k | a_k < a_i\}$, then $v_1[p] \geq v_1[\ell]$;*
2. *$v_1, v_2 \in \{a_k | a_k \geq a_i\}^*$, and for $\ell < p$, if $v_2[p], v_2[\ell] \in \{a_k | a_k > a_j\}$, then $v_2[p] \geq v_2[\ell]$, and if $v_1[p], v_1[\ell] \in \{a_k | a_k > a_j\}$, then $v_1[p] \leq v_1[\ell]$.*

Proof. First we consider the *only if* direction. Clearly, if both $v_1, v_2 \in \{a_k | a_i \leq a_k \leq a_j\}^*$, then $w = a_ia_jv_1a_ja_ia_ja_iv_2a_ia_j$ is amiable to $w' = a_ja_iv_1a_ia_ja_ia_jv_2a_ja_i$, from Proposition 20 and Remark 21.

If only one of v_1 and v_2 is not in $\{a_k | a_i \le a_k \le a_j\}^*$, then from Proposition 20 and Remark 21, we know that w and w' are not amiable under any $\mathbb{P}$-Parikh matrix.

Let us consider the nontrivial case when neither v_1 nor v_2 are in the set $\{a_k | a_i \le a_k \le a_j\}^*$. Recall that we use $|v|_u$ to denote the number of distinct occurrences of u as a subword in v.

Suppose there exists a letter a_ℓ where $a_\ell \notin \{a_k | a_i \le a_k \le a_j\}$, and $|v_1|_{a_\ell} \neq |v_2|_{a_\ell}$. If $a_j < a_\ell$, then $|w|_{a_i a_j a_\ell} - |w'|_{a_i a_j a_\ell} = |v_1|_{a_\ell} + 4 \times |v_2|_{a_\ell} - 5 \times |v_2|_{a_\ell} = |v_1|_{a_\ell} - |v_2|_{a_\ell}$. It follows that $|w|_{a_i a_j a_\ell} = |w'|_{a_i a_j a_\ell}$ if and only if $|v_1|_{a_\ell} = |v_2|_{a_\ell}$. If $a_\ell < a_i$, the same conclusion can be reached.

Now, suppose both v_1 and v_2 contain letters a_k, a_ℓ where $a_k < a_i$ and $a_\ell > a_j$ (notice that we require the number of a_k's and a_ℓ's to be the same in both v_1 and v_2, which is why we only consider the case where both letters are in both subwords). Then,

$$|w|_{a_k a_i a_j a_\ell} - |w'|_{a_k a_i a_j a_\ell} = |v_1|_{a_k} \times |v_2|_{a_\ell} - 3 \times |v_1|_{a_k} \times |v_2|_{a_\ell} = -2 \times |v_1|_{a_k} \times |v_2|_{a_\ell}.$$

So, v_1 and v_2 may only contain letters that are *all* less than or equal to a_j, or *all* greater than or equal to a_i.

Finally, we must confirm that any subword count containing multiple letters on one side of a_i and a_j are equal. Let us start with a subword counting 2 letters on one side of a_i, a_j. Let $a_{k_1}, a_{k_2} \in \{a_k | a_j < a_k\}$, and let $S = \{a_i, a_j, a_{k_1}, a_{k_2}\}$ (the case where $a_{k_1}, a_{k_2} \in \{a_k | a_k < a_i\}$ is quite similar). Then, by denoting $|w|_S$, for a word w and set S, as the number of occurrences in w of subwords permitted by the set S and the order of the alphabet, we have

$$\begin{aligned} |w|_S - |w'|_S &= |v_1|_{\{a_{k_1}, a_{k_2}\}} + 4 \times |v_2|_{\{a_{k_1}, a_{k_2}\}} + |v_1|_{a_{k_1}} \times |v_2|_{a_{k_2}} - 5 \times |v_2|_{\{a_{k_1}, a_{k_2}\}} \\ &= |v_1|_{\{a_{k_1}, a_{k_2}\}} + |v_1|_{a_{k_1}} \times |v_2|_{a_{k_2}} - |v_2|_{\{a_{k_1}, a_{k_2}\}}. \end{aligned} \tag{2}$$

Clearly we have that $|u|_{\{a_i, a_j\}} \le |u|_{a_i} \times |u|_{a_j}$. Since $|v_1|_{a_{k_1}} = |v_2|_{a_{k_1}}$, and $|v_1|_{a_{k_2}} = |v_2|_{a_{k_2}}$, we have that $|v_1|_{a_{k_1}} \times |v_2|_{a_{k_2}}$ is the maximum possible value for both $|v_1|_{\{a_{k_1}, a_{k_2}\}}$ and $|v_2|_{\{a_{k_1}, a_{k_2}\}}$. So, from Equation (2), we need $|v_2|_{\{a_{k_1}, a_{k_2}\}} - |v_1|_{\{a_{k_1}, a_{k_2}\}} = |v_1|_{a_{k_1}} \times |v_2|_{a_{k_2}}$. Since both $|v_1|_{\{a_{k_1}, a_{k_2}\}}$ and $|v_2|_{\{a_{k_1}, a_{k_2}\}}$ must be non-negative, and $|v_1|_{a_{k_1}} \times |v_2|_{a_{k_2}}$ is the maximum possible value for both $|v_1|_{\{a_{k_1}, a_{k_2}\}}$ and $|v_2|_{\{a_{k_1}, a_{k_2}\}}$, in order to satisfy the equation, we need $|v_2|_{\{a_{k_1}, a_{k_2}\}} = |v_1|_{a_{k_1}} \times |v_2|_{a_{k_2}}$ and $|v_1|_{\{a_{k_1}, a_{k_2}\}} = 0$. This occurs exactly when all a_{k_1} precede all a_{k_2} in v_2, and all a_{k_2} precede all a_{k_1} in v_1.

Now we must prove the *if* direction. Moreover, we show that for any length subword, if the given word w satisfies either Condition 1 or Condition 2, then w' is amiable to w. Let $a_{k_1}, a_{k_2}, \ldots, a_{k_m} \in$

$\{a_k|a_j < a_k\}$ and $S = \{a_i, a_j, a_{k_1}, a_{k_2}, \ldots, a_{k_m}\}$. The case where $a_{k_1}, a_{k_2}, \ldots, a_{k_m} \in \{a_k|a_k < a_i\}$ is similar. Then we have $|w|_S - |w'|_S = |v_1|_{a_{k_1}} \times |v_2|_{S\setminus\{a_i,a_j,a_{k_1}\}} + 4 \times |v_2|_{S\setminus\{a_i,a_j\}} - 5 \times |v_2|_{S\setminus\{a_i,a_j\}}$. However, we know that $|v_1|_{a_{k_1}} = |v_2|_{a_{k_1}}$ and since all a_{k_1} precede all other a_{k_p} in v_2 we can hence conclude that $|v_1|_{a_{k_1}} \times |v_2|_{S\setminus\{a_i,a_j,a_{k_1}\}} = |v_2|_{S\setminus\{a_i,a_j\}}$. Therefore we have $|w|_S - |w'|_S = 5 \times |v_2|_{S\setminus\{a_i,a_j\}} - 5 \times |v_2|_{S\setminus\{a_i,a_j\}} = 0$. □

In other words, the above statement says that two words are not $\mathbb{P}$-distinct if both v_1 and v_2 are defined on the subset of the alphabet which is either lexicographically bigger than a_i or smaller than a_j, and they share the same Parikh vector for the subset of letters which are not in between a_i and a_j. Furthermore, if $v_1 \in \{a_{i+1}, a_{i+2}, \ldots, a_n\}^*$, then all the letters which are lexicographically greater than a_j must occur in v_1 in decreasing lexicographical order and in v_2 in increasing order. On the other hand, if $v_1 \in \{a_1, a_2, \ldots, a_{j-1}\}^*$, then all the letters which are lexicographically smaller than a_i must occur in v_1 in lexicographical order and in v_2 in decreasing lexicographical order.

4. $\mathbb{L}$-Parikh Matrices

Proposition 17 shows that, in many cases, the set of words that share both a Parikh matrix and a $\mathbb{P}$-Parikh matrix (with respect to a certain alphabet and subset S of that alphabet) is smaller than the set of those that share only a Parikh matrix. However, following Corollary 19, we also know that this never happens for binary alphabets. Hence we now study $\mathbb{L}$-Parikh matrices as an alternative method of ambiguity reduction.

We focus on binary alphabets, although $\mathbb{L}$-Parikh matrices can also be effective for any non-unary alphabet, as illustrated by the following example:

Example 23. *Consider the word $w_1 = abcda$ from Example 3, which is amiable with the words $w_2 = abcad$, $w_3 = abacd$ and no others. We then observe that:*

$$\Psi_L(w_1) = \Psi(aabcd) = \left\langle \begin{smallmatrix} 2 & 2 & 2 & 2 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{smallmatrix} \right\rangle \quad \Psi_L(w_2) = \Psi(abcad) = \left\langle \begin{smallmatrix} 2 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{smallmatrix} \right\rangle \quad \Psi_L(w_3) = \Psi(abacd) = \left\langle \begin{smallmatrix} 2 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{smallmatrix} \right\rangle.$$

Therefore w_1 has a different $\mathbb{L}$-Parikh matrix than w_2 and w_3, and we can in fact uniquely describe w_1 using a combination of its Parikh matrix and $\mathbb{L}$-Parikh matrix.

We begin this section by explaining the motivation for choosing the Lyndon conjugate of a word and then work towards our main result where we characterise words whose ambiguity is reduced by the use of $\mathbb{L}$-Parikh matrices.

As indicated by Definition 8, the concept of $\mathbb{L}$-Parikh matrices is based on a modification to the word that results in a change in the order of letters. The following theorem implies that the strategy of altering a word is not always a successful method of ambiguity reduction. Note that Ψ_{rev} refers to the Parikh matrix of the reversal of a word.

Theorem 24 ([10]). *For any word w, $\mathcal{A}(w, \Psi) = \mathcal{A}(w, \Psi, \Psi_{\mathrm{rev}})$.*

Unlike Theorem 24, $\mathbb{L}$-Parikh matrices use the conjugate of a word. The next proposition implies that such conjugates need to be chosen wisely.

Proposition 25. *Consider two words $v, w \in \Sigma^*$ with $\Psi(v) = \Psi(w)$. For any factorisations $v = v_1 v_2$ and $w = w_1 w_2$ such that $|v_2| = |w_2|$ and one of the matrices $\Psi(v), \Psi(v_2 v_1), \Psi(w_2 w_1)$ contains at least one non-zero element that is not on the diagonal, we have that $\Psi(v_2 v_1) = \Psi(w_2 w_1)$ implies $\phi(v_2) = \phi(w_2)$. In the case of binary alphabets, when Σ_2 is considered, the reverse direction also stands, namely $\phi(v_2) = \phi(w_2)$ implies $\Psi(v_2 v_1) = \Psi(w_2 w_1)$.*

Proof. First we consider the statement that applies to any size alphabet.

We know that $\Psi(v_2 v_1) = \Psi(w_2 w_1)$ and must prove that $\phi(v_2) = \phi(w_2)$. This is done by contradiction. Let us assume that $\Psi(v) = \Psi(w)$ such that $\Psi(v_2 v_1) = \Psi(w_2 w_1)$ and $\phi(v_2) \neq \phi(w_2)$.

Since $\phi(v_2) \neq \phi(w_2)$, there is at least one letter in the alphabet Σ for which the number of times it occurs in v_2 is not equal to the number of times it can be found in w_2. We assume without loss of generality that the first letter this inequality occurs for is a. For any other cases, we just need to consider the respective letter and the one following it. Observe that this situation is not possible for the last letter of the alphabet, as in that case the contradiction is immediate.

We now consider the subword ab and find the number of occurrences of it as a subword in v and w, and in $v_2 v_1$ and $w_2 w_1$, respectively,

$$
\begin{aligned}
\Psi(v) = \Psi(w) &\implies |v|_{ab} = |w|_{ab} \\
&\iff |v_1|_{ab} + |v_2|_{ab} + |v_1|_a |v_2|_b = |w_1|_{ab} + |w_2|_{ab} + |w_1|_a |w_2|_b \qquad (3) \\
\Psi(w_2 w_1) = \Psi(v_2 v_1) &\implies |w_2 w_1|_{ab} = |v_2 v_1|_{ab} \\
&\iff |w_1|_{ab} + |w_2|_{ab} + |w_2|_a |w_1|_b = |v_1|_{ab} + |v_2|_{ab} + |v_2|_a |v_1|_b. \qquad (4)
\end{aligned}
$$

Now we add Equations (3) and (4) together to obtain

$$|v_1|_a|v_2|_b + |w_2|_a|w_1|_b = |w_1|_a|w_2|_b + |v_2|_a|v_1|_b. \tag{5}$$

We again assume without loss of generality that $|v_2|_a > |w_2|_a$.

There are three possible cases for the relationship between $|v_1|_b$ and $|w_1|_b$. Either $|v_2|_b = |w_2|_b$, $|v_2|_b < |w_2|_b$ or $|v_2|_b > |w_2|_b$.

First we assume that $|v_2|_b = |w_2|_b$. Thus we also have that $|v_1|_b = |w_1|_b$, since the Parikh matrices for v and w are equal. If the equality stands, then we can rewrite Equation (5) as follows, with $x = |v_2|_b = |w_2|_b$ and $y = |v_1|_b = |w_1|_b$,

$$x|v_1|_a + y|w_2|_a = x|w_1|_a + y|v_2|_a.$$

Since $|v_2|_a > |w_2|_a$ and $\Psi(v) = \Psi(w)$, we must also have that $|v_1|_a < |w_1|_a$. This means that both remaining components on the left hand side of the equality are less than those on the right hand side. Therefore we reach a contradiction and our initial assumption that $|v_2|_b = |w_2|_b$ is false.

Next assume $|v_2|_b < |w_2|_b$. Then

$$\Psi(v) = \Psi(w) \implies |v_1|_a + |v_2|_a = |w_1|_a + |w_2|_a \implies |v_1|_a < |w_1|_a,$$
$$\Psi(v) = \Psi(w) \implies |v_1|_b + |v_2|_b = |w_1|_b + |w_2|_b \implies |v_1|_b > |w_1|_b.$$

If we now apply all of these inequalities to Equation (5), we find that all components on the left hand side of the equality are less than those on the right hand side, and the equality cannot be true. Hence also our assumption that $|v_2|_b < |w_2|_b$ is false.

The only remaining possibility is that $|v_2|_b > |w_2|_b$. In this case we can apply the same method to the subword bc and we eventually obtain the inequality $|v_2|_c > |w_2|_c$. If we repeat the process iteratively for all subwords of length 2 that are admitted by the order of the alphabet, we find for every letter $e \in \Sigma$ that $|v_2|_e > |w_2|_e$. But following the Proposition's statement we have $|v_2| = |w_2|$. We therefore get a contradiction and our initial assumption that $\Psi(v) = \Psi(w)$ and $\Psi(v_2v_1) = \Psi(w_2w_1)$ and $\phi(v_2) \neq \phi(w_2)$ was false.

We can therefore conclude that whenever $\Psi(v_1v_2) = \Psi(w_1w_2)$ and $\Psi(v_2v_1) = \Psi(w_2w_1)$, we have that $\phi(v_2) = \phi(w_2)$.

Now for the statement that only applies to a binary alphabet. We prove this through contradiction by assuming that $\phi(v_2) = \phi(w_2)$, $\Psi(v_1v_2) = \Psi(w_1w_2)$ and $\Psi(v_2v_1) \neq \Psi(w_2w_1)$.

If $\Psi(v_2v_1) \neq \Psi(w_2w_1)$ and $\Psi(v_1v_2) = \Psi(w_1w_2)$, since $\phi(v_2) = \phi(w_2)$ we have $|v_2v_1|_{ab} \neq |w_2w_1|_{ab}$. Furthermore,

$$\begin{aligned} |v_2v_1|_{ab} \neq |w_2w_1|_{ab} &\iff |v_2|_{ab} + |v_1|_{ab} + |v_2|_a|v_1|_b \neq |w_2|_{ab} + |w_1|_{ab} + |w_2|_a|w_1|_b \\ &\implies |v_2|_{ab} + |v_1|_{ab} \neq |w_2|_{ab} + |w_1|_{ab}. \end{aligned}$$

But from the assumption that $\Psi(v_1v_2) = \Psi(w_1w_2)$, we have that

$$|v_2|_{ab} + |v_1|_{ab} + |v_1|_a|v_2|_b = |w_2|_{ab} + |w_1|_{ab} + |w_1|_a|w_2|_b,$$

and because $\phi(v_2) = \phi(w_2)$, which implies that $\phi(v_1) = \phi(w_1)$, we have that $|v_2|_{ab} + |v_1|_{ab} = |w_2|_{ab} + |w_1|_{ab}$. Hence we have a contradiction and our initial assumption that when $\phi(v_2) = \phi(w_2)$ and $\Psi(v_1v_2) = \Psi(w_1w_2)$, we get that $\Psi(v_2v_1) \neq \Psi(w_2w_1)$ is false. This concludes our proof. $\square$

Below is an example that shows the significance of the condition $|v_2| = |w_2|$ in Proposition 25.

Example 26. *Consider the words $v = baabbaaaaabbbb$ with $v_2 = aaaaabbbb$ and $w = aabaabbbbaaabb$ with $w_2 = aaabb$. One can easily find that $\Psi(v_2v_1) = \Psi(w_2w_1) = \left\langle \begin{smallmatrix} 7 & 39 \\ & 7 \end{smallmatrix} \right\rangle$. Furthermore, we have that $\Psi(v) = \Psi(w)$, $\Psi(v_2v_1) = \Psi(w_2w_1)$ and $|v_2| \neq |w_2|$. However $\phi(v_2) \neq \phi(w_2)$, since $\phi(v_2) = [5, 4]$ and $\phi(w_2) = [3, 2]$, and therefore $|v_2| = |w_2|$ is important in the context of Proposition 25.* $\triangleleft$

We also show through an example the importance of the condition that at least one of the matrices $\Psi(v), \Psi(v_2v_1), \Psi(w_2w_1)$ contains at least one non-zero element that is not on the diagonal in Proposition 25.

Example 27. *Consider the words $v = ac$ with $v_2 = c$ and $w = ca$ with $w_2 = a$. Clearly $|v_2| = |w_2|$ and $\phi(v_2) \neq \phi(w_2)$. However, $\Psi(v) = \Psi(w) = \Psi(v_2v_1) = \Psi(w_2w_1) = \left\langle \begin{smallmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{smallmatrix} \right\rangle$. This is due to the lack of any lexicographically adjacent letters in any of the four words. Hence the property that at least one non-zero element that is not on the diagonal of at least one of the three Parikh matrices is an essential one for Proposition 25.*

An example for the ternary alphabet where $\Psi(v_2v_1) \neq \Psi(w_2w_1)$ even though we have that $\Psi(v) = \Psi(w)$ and $\phi(v_2) = \phi(w_2)$ is given below. Note that if $\phi(v_2) = \phi(w_2)$, then $|w_2| = |v_2|$.

Example 28. *Consider the words $v = cbbaaabb$ with $v_2 = aabb$ and $w = cabbbaab$ with $w_2 = baab$. We have $\Psi(v) = \Psi(w)$ and, moreover, $|w_2| = |v_2|$ and $\phi(v_2) = \phi(w_2)$. Nevertheless, we have in this case that $\Psi(v_2v_1) = \Psi(aabbcbba) \neq \Psi(baabcabb) = \Psi(w_2w_1)$.* $\triangleleft$

Since the statement that applies to binary alphabets from Proposition 25 does not hold for ternary alphabets, we conclude that in fact it does not hold for any alphabet of size greater than 2.

Furthermore, by inspecting the case where $|v_2|_b > |w_2|_b$ in the proof of Proposition 25, we make the following remark regarding the binary case.

Remark 29. *Consider two binary words $v, w \in \Sigma^*$ with $\Psi(v) = \Psi(w)$. If $v = v_1v_2$ and $w = w_1w_2$ with $\Psi(v_2v_1) = \Psi(w_2w_1)$, then either $\phi(v_2) = \phi(w_2)$, or $|v_2|_a \geq |w_2|_a$ and $|v_2|_b > |w_2|_b$.*

Proposition 25 shows that when looking for a modification that we can apply to a word to find a new and different Parikh matrix, we need to consider conjugates of amiable words where it is less likely that the Parikh vectors of their right factors are the same, i. e., conjugates obtained by shifting the original words a different number of times, respectively.

Let us now consider how using $\mathbb{L}$-Parikh matrices reduces ambiguity. Throughout the rest of this section we do not consider words that are uniquely described by their Parikh matrix, i.e., we ignore any word w where $|\mathcal{A}(w, \Psi)| = 1$, since there is no ambiguity to be reduced here. For a word w, we calculate $\Psi(w)$ and $\Psi_L(w)$ and use both of these matrices to describe the original word. The ambiguity of a word w, with respect to its Parikh and $\mathbb{L}$-Parikh matrices, according to Definition 5, is the total number of words that share a Parikh matrix and an $\mathbb{L}$-Parikh matrix with w, namely $|\mathcal{A}(w, \Psi, \Psi_L)|$. Therefore, a reduction in ambiguity means that $|\mathcal{A}(w, \Psi)| > |\mathcal{A}(w, \Psi, \Psi_L)|$.

We use the next definitions and propositions to build towards our main result where we characterise binary words whose ambiguity is reduced using $\mathbb{L}$-Parikh matrices. In line with Definition 15 we introduce the following notions.

Definition 30. *Given a word $w \in \Sigma^*$, we call $\Psi(w)$* $\mathbb{L}$-distinguishable *if either $|\mathcal{A}(w, \Psi)| = 1$ or there exists a word $u \in \Sigma^*$ with $\Psi(w) = \Psi(u)$, such that $\Psi_L(w) \neq \Psi_L(u)$. In the latter case we say that w and u are* $\mathbb{L}$-distinct*. A word w is* $\mathbb{L}$-unique *if $|\mathcal{A}(w, \Psi, \Psi_L)| = 1$.*

Observe that if w and v are $\mathbb{L}$-*distinct*, then $\mathcal{A}(w, \Psi) = \mathcal{A}(v, \Psi)$ and $\mathcal{A}(w, \Psi, \Psi_L) \neq \mathcal{A}(v, \Psi, \Psi_L)$. The example below demonstrates the effectiveness of $\mathbb{L}$-Parikh matrices for ambiguity reduction.

Example 31. *Consider the words $w = babbbaa$, $u = bbababa$ and $v = bbbaaab$ with $\Psi(w) = \Psi(u) = \Psi(v)$. When considering the conjugates $L(w) = aababbb$, $L(u) = abababb$ and $L(v) = aaabbbb$ we have that $\Psi_L(w) = \langle {}^{3\ 11}_{\ \ 4} \rangle$, $\Psi_L(u) = \langle {}^{3\ 9}_{\ \ 4} \rangle$, and $\Psi_L(v) = \langle {}^{3\ 12}_{\ \ 4} \rangle$. Thus their $\mathbb{L}$-Parikh matrices are all different and we can uniquely describe each of the words by using its Parikh matrix and $\mathbb{L}$-Parikh matrix.* ◁

$\mathbb{L}$-distinguishability is a necessary property for ambiguity reduction with $\mathbb{L}$-Parikh matrices.

Corollary 32. *Consider a word* $w \in \Sigma^*$. *Then* $|\mathcal{A}(w, \Psi)| > |\mathcal{A}(w, \Psi, \Psi_L)|$ *if and only if* $\Psi(w)$ *is* $\mathbb{L}$-*distinguishable.*

The above characterisation of ambiguity reduction leads us to investigate sufficient conditions for a matrix to be ambiguous, and therefore for any pair of words it describes not to be $\mathbb{L}$-distinct. Our next results consider the situations when the Parikh matrix of a word is not $\mathbb{L}$-distinguishable. We demonstrate that the occurrence of words that meet the criteria outlined in each proposition is rare either later in the paper or directly following the next proposition.

Proposition 33. *For a word* $w \in \Sigma^*$, *if all words in* $\mathcal{A}(w, \Psi)$ *belong to the same conjugacy class, then* $\Psi(w)$ *is not* $\mathbb{L}$-*distinguishable.*

This proposition is a direct consequence of Definition 30.

Example 34. *Let* $w = aababa$ *and* $w' = abaaab$. *These two words are amiable with each other and nothing else. Furthermore,* $L(w) = aaabab = L(w')$, *and since both words share a Lyndon conjugate, both words also share the same* $\mathbb{L}$-*Parikh matrix. Therefore* $\Psi(w)$ *is not* $\mathbb{L}$-*distinguishable.* ◁

Now we move on to explore, for binary alphabets, the case where all words in $\mathcal{A}(w, \Psi)$ belong to the same conjugacy class in more detail. Recall that $C(w)$ refers to the conjugacy class of w.

Proposition 35. *Consider a binary word* $w \in \Sigma_2^*$. *For all* $u \in \mathcal{A}(w, \Psi)$, *we have* $L(u) = L(w)$ *if and only if* $C(w)$ *is closed under* Type 2 *transformations.*

Proof. First consider the *if* direction where we have that $C(w)$ is closed under Type 2 transformations. This means that applying a Type 2 transformation to any word in $C(w)$ results in a conjugate of the original word, namely a word which is also in $C(w)$. We know that Type 2 transformations classify binary words that share a Parikh matrix. Hence the Parikh matrix, $\Psi(w)$, of any word in $C(w)$ only describes other words in $C(w)$. Therefore, all words associated to $\Psi(w)$ are members of the same conjugacy class and hence share the same Lyndon conjugate.

Now consider the *only if* direction. We have that $L(u) = L(w)$ for all $u \in \mathcal{A}(w, \Psi)$. Consider two words $v = xabybaz$ and $v' = xbayabz$, where $v, v' \in \mathcal{A}(w, \Psi)$ and $x, y, z \in \Sigma_2^*$. Since Type 2 transformations characterise binary alphabets, we know that if w is unambiguous then such words exist and v' can be obtained from v through Type 2 transformations, and vice versa. Also, since $v, v' \in \mathcal{A}(w, \Psi)$ and therefore $L(v) = L(v')$, we have that v and v' are conjugates.

Now consider the possible conjugates of v and v'. Let us take all possible factorisations $x = x_1x_2$, $y = y_1y_2$ and $z = z_1z_2$, respectively. Since $\Psi(v) = \Psi(v')$ we have, for all $0 \le |x_1| \le |x|$, that

$$\Psi(x_2abybazx_1) = \Psi(x_2bayabzx_1). \tag{6}$$

For all $0 \le |y_1| \le |y|$ we have that

$$\Psi(y_2bazxaby_1) = \Psi(y_2abzxbay_1). \tag{7}$$

Finally, for all $0 \le |z_1| \le |z|$ we have that

$$\Psi(z_2xabybaz_1) = \Psi(z_2xbayabz_1). \tag{8}$$

It is not difficult to see that all of the words described in Equations (6), (7) and (8) are from the same conjugacy class as v and v', namely $C(w)$ according to our hypothesis.

The only other words from the respective conjugacy class are: $bybazxa, ayabzxb, azxabyb$ and $bzxbaya$. Clearly we cannot apply a Type 2 transformation here, as one of the ab or ba factors we wish to use for the transformation are split across the start/end of the word. Hence we ignore these cases as they are considered for different values of v and v'.

Therefore every conjugate of v and v', which are conjugates of w themselves, can be transformed through Type 2 transformations to obtain other conjugates of w. Note that every word in $\mathcal{A}(w, \Psi)$ can be obtained from another word in $\mathcal{A}(w, \Psi)$ through Type 2 transformations. Hence by applying this process to every word and its transformed counterpart, and thereby covering every possible occurrence of a Type 2 transformation that exists in every conjugate of w, we find that only conjugates of w can be obtained by applying Type 2 transformations to conjugates of w. We conclude that $C(w)$ is closed under Type 2 transformations. □

We now prove a property that is true for all words whose conjugacy class is closed under Type 2 transformations.

Proposition 36. *Consider a binary word $w \in \Sigma_2^*$. Then $C(w)$ is closed under* Type 2 *transformations if and only if $L(w) \in \{aabb, ababbb, aababb, aabbab, aaabab\}$.*

Proof. The *if* direction is easily proven by checking the property for every word in the conjugacy class of the words in the set $A = \{aabb, ababbb, aababb, aabbab, aaabab\}$.

Let us define a *block of a letter* to be a unary factor of a word which is not extendable to the right or the left. For the proof of the *only if* direction, in order to show that a conjugacy class is closed under Type 2 transformations, we need to show that, in a given word, there are no positions to which we can apply a Type 2 transformation that create a new word that is not a conjugate of the original. We can thus restrict ourselves to only Lyndon conjugates of each class from now on, and prove this statement by therefore considering maximal blocks of a's at the start of our words.

In particular, we show that a conjugacy class is not closed under Type 2 transformations by identifying a position in a word that we can apply a Type 2 transformation to such that it produces a word that is not a conjugate of the original. That is, the transformation either alters the size of the initial block of a's or changes the number of blocks of a's within the word, and therefore must create a word that is not a conjugate of the original. We show that this is possible for all binary words except for those in the predifined set A.

Let n be the number of consecutive a's at the start of the word and m be the number of consecutive b's that immediately follow the first block of a's. Since we only consider Lyndon conjugates, we know that n is the largest number of consecutive a's in the word. Moreover, we assume that every word that we consider has at least 2 a's and 2 b's, to ensure that a Type 2 transformation can always be applied to at least one conjugate of that word.

To begin, consider Lyndon words with only one block of a's, namely words of the form $a^n b^m$. Applying a Type 2 transformation to any conjugate $ba^n b^{m-1}$ of our word, we create a word with 2 blocks of a's, namely

$$\boldsymbol{ba}a^{n-2}\boldsymbol{ab}b^{m-2} \rightarrow \boldsymbol{ab}a^{n-2}\boldsymbol{ba}b^{m-2},$$

unless $n = m = 2$, which is the first element in our exceptions set, $aabb \in A$.

Now consider the case where a word has two blocks of a's. Let us define positive integers s and t, and therefore investigate words of the form $a^n b^m a^s b^t$ with $n \geq s$.

First consider the case where $n = 1$. When $m = 1$, our word must be of the form $abab^t$ with $t > 1$, since we only consider Lyndon conjugates and we cannot apply a Type 2 transformation to any conjugate of the word $abab$. Consider the Type 2 transformation applied to one of its conjugates

$$abab^t \rightarrow \boldsymbol{ba}b\boldsymbol{ab}b^{t-2} \rightarrow abbbab^{t-2}.$$

Clearly these words are conjugates only for $t = 3$, which is our second exception, namely $ababbb \in A$.

When $m \geq 2$ we consider the Lyndon conjugate $ab^m ab^{\geq m}$. Applying the Type 2 transformation

$$\boldsymbol{ab}b^{m-2}\boldsymbol{ba}b^{\geq m} \to bab^{m-2}abb^{\geq m},$$

allows us to obtain a word that is not a conjugate of the original since now the size of the shortest block of b's in the word is $m - 2 < m$ (every block of b's follows a single a).

Now consider the case when $n \geq 2$ and the application of the following Type 2 transformation

$$a^n b^m a^s b^t \to \boldsymbol{ba}a^{n-2}\boldsymbol{ab}b^{m-1}a^s b^{t-1} \to aba^{n-2}bab^{m-1}a^s b^{t-1}.$$

Let us consider the cases when this transformation results in conjugates of $a^n b^m a^s b^t$. If $n = 2$ and $n = 3$ we obtain that the only words that remain conjugates after the transformation are $aabbab, aaabab$ and $aababb$. All of these words are included in our predefined set A. Moreover, for $n \geq 4$, we find that we obtain a conjugate after this transformation also when $n = s + 2$, $m = 1$ and $t = 1$. If we substitute these values into the above transformation, we have that

$$a^n ba^{n-2}b \to aba^{n-2}baa^{n-2}.$$

However, in this case when we consider applying yet another Type 2 transformation

$$a\boldsymbol{ba}a^{n-4}\boldsymbol{ab}aa^{n-2} \to aaba^{n-4}baaa^{n-2}$$

we obtain a word that has two blocks of a's, one of size $n - 4$ and one of size $n + 2$. However, since n is the largest block of a's in the original word we obtain contradictions in the case of all binary words with 2 blocks of a's and $n \geq 4$.

In the case of 3 or more blocks, we apply the Type 2 transformation by using the final a in the last block and the b that follows it to be our ab, and choose the first a in the second block and the b that precedes it to be our ba. Please note that in all such cases either the size of the first block of a's increases, or it remains the same, but in this case the size of the block of b's following it decreases. In the latter of the cases, we also have an increase of the number of blocks of a's in the word. Both of these situations lead to the construction of a word that is lexicographically smaller than the word we have started with. This concludes the proof. □

By Propositions 35 and 36, the following assertion holds:

Proposition 37. *Consider a binary word $w \in \Sigma_2^*$. Then $L(u) = L(w)$, for all $u \in \mathcal{A}(w, \Psi)$, if and only if $L(w) \in \{aabb, ababbb, aababb, aabbab, aaabab\}$.*

We now look at the case where all words associated to a Parikh matrix are the Lyndon representatives of their respective conjugacy classes, which again makes this matrix not $\mathbb{L}$-distinguishable.

Proposition 38. *Consider a word $w \in \Sigma^*$. If $|\mathcal{A}(w, \Psi)| \geq 2$ and for every $u \in \mathcal{A}(w, \Psi)$ we have $u = L(u)$, then $\Psi(w)$ is not $\mathbb{L}$-distinguishable.*

Proof. Since for every u we have that $u = L(u)$, we also have that $\Psi(u) = \Psi_L(u) = \Psi(w)$ for all $u \in \mathcal{A}(w, \Psi)$. Hence there exists no word v where $\Psi(v) = \Psi(w)$, and $\Psi_L(v) \neq \Psi_L(w)$. Therefore $\Psi(w)$ is not $\mathbb{L}$-distinguishable. □

An example of this situation is described below.

Example 39. *The words $w = aaaabaabbb$ and $w' = aaaaabbabb$ are only amiable with each other and both are the Lyndon representatives of their respective conjugacy classes. Therefore, $\Psi(w) = \Psi(w') = \Psi_L(w) = \Psi_L(w')$ and $\Psi(w)$ is not $\mathbb{L}$-distinguishable.* ◁

For binary alphabets, we examine in greater detail when all words in $\mathcal{A}(w, \Psi)$ are the Lyndon representatives of their conjugacy classes. The next result provides a necessary and sufficient condition, and therefore the complete characterisation, for this case to occur for binary words.

Proposition 40. *Consider a binary word $w \in \Sigma_2^*$. Then the following statements are equivalent.*

- *For all $u \in \mathcal{A}(w, \Psi)$, we have that $u = L(u)$.*
- *$w = a^*vb^*$ and for $n = |v|_{ba}$, we have that $|v|_a = 2n$ and $|v|_b = n + 1$.*

Proof. Let us initially consider words of the form $w = a^*vb^*$ where for $n = |v|_{ba}$ we have $|v|_a = 2n = x$ and $|v|_b = n + 1 = y$. For this direction, we first show that $w = L(w)$, then move on to show that only words of the form a^*vb^* are associated to $\Psi(w)$.

If $w = v$, since $|v|_a = 2n$ and $|v|_{ba} = n$, then there must be a minimum of n occurrences of a's before the first b, otherwise $|v|_{ba} > n$. Furthermore, since $|v|_b = n + 1$ and $|v|_{ba} = n$ there must be a minimum of one b after the last a.

Let us now consider the factor z that begins with the first occurrence of b and ends with the last occurrence of a in v. Then $|v|_{ba} = |z|_{ba} = n$, $|z|_a \leq n$ and $|z|_b \leq n$. Hence $|z| \leq 2n$.

For a word to be the Lyndon conjugate of its class, the block of a's at the start of the word must be greater than or equal to any other block of a's in the word, and the word must end with at least one b. We already know that v ends with at least one b so let us now consider the a's at the start of v. There are two cases for the number of a's here. Note that since there are at most n occurrences of a in z, there must be at least n occurrences of a before z. We know that there are no a's after z since, by definition, it ends with the last occurrence of a in v. If there are more than n consecutive a's at the start of v, then $|z|_a < n$ and we clearly have that $v = L(v)$. Otherwise, there are exactly n consecutive a's at the start of v. This implies that $|z|_a = n$, from $|v|_a = 2n$. Therefore, since $|z|_{ba} = n$, we have that $|z|_b = 1$. By definition of z, this b is the first letter, so the n occurrences of a must be consecutive. Hence $v = a^n ba^n b^n$, which clearly is a Lyndon representative of its conjugacy class.

Next consider a word of the form $w = a^* vb^*$. If we add any number of a's to the start of a word that is already the Lyndon conjugate of its class, then that word remains the Lyndon conjugate of some class. The same holds for adding any number of b's to the end of a word that is the Lyndon conjugate of its class. Hence $w = a^* vb^* = L(w)$.

We showed that if we have a word $w = a^* vb^*$ such that $n = |v|_{ba}$ with $|v|_a = 2n$ and $|v|_b = n+1$, then w is the Lyndon representative of its conjugacy class. Next we show that if a word has this form, then it is amiable only with words that can also be written as such.

Let w' be a word that is amiable with w. We know that a series of Type 2 transformations can be applied to w to obtain w'. Our aim is to show that this series of transformations can only be applied to w in such a way that w' can always be written in the same way as w.

Note that since $\Psi(w) = \Psi(w')$, we conclude from Theorem 24 that $|w|_{ba} = |w'|_{ba} = n$. Moreover, since $|w|_a = |w'|_a$ and $|w|_b = |w'|_b$, we can rewrite w and w' as

$$w = a^r vb^s \qquad \text{and} \qquad w' = a^r v' b^s,$$

where for $|v|_{ba} = |v'|_{ba} = n$, we have that $|v|_a = |v'|_a = 2n$ and $|v|_b = |v'|_b = n + 1$. If this was not the case for w', then either w' would have a b in the first r positions, or an a in the last s positions. However, in both of these cases contradictions would be reached with respect to the total number of ba's in the word, as $|w'|_a = 2n + r$ and $|w'|_b = n + s + 1$.

Following our above reasoning in the case of w, since $w' \in a^* vb^*$, we have that also $w' = L(w')$, which concludes the proof for the first direction.

Now assume that the words $w_1, w_2, \ldots, w_k \in \mathcal{A}(w, \Psi)$ are pairwise distinct Lyndon conjugates, where $|\mathcal{A}(w, \Psi)| = k$. Without loss of generality, we assume that $w_i <_{lex} w_j$, for $1 \leq i < j \leq k$, and denote $|w_i|_{ba} = m$. To prove this direction, we enumerate all words of this form and describe the properties which these words must have. Furthermore, note that words of the form a^*b^* are uniquely described by their respective Parikh matrices, and comply with our description, hence we do not include these in our subsequent search for words that are associated to matrices that only describe Lyndon representatives of their conjugacy classes.

We start by observing that since $|\mathcal{A}(w, \Psi)| = k$ and w_1 is the smallest lexicographical element of the set $\mathcal{A}(w, \Psi)$, then $w_1 \in a^j b^m a b^\ell$ with $j, \ell > 0$. Indeed, since $|w_1|_{ba} = m$ and our word has at least two blocks of a's, if the first block of b's is of size greater than m or the second block of a's is of size greater than 1, then we get a lexicographically smaller word through a Type 2 transformation.

In the same way, we can now look at w_k, the largest lexicographical word in $\mathcal{A}(w, \Psi)$. It is straightforward that, since $|w_k|_{ba} = m$, following the above reasoning, $w_k \in a^{j-m+1} b a^m b^{\ell+m-1}$ with $j - m + 1 \geq m$. However, this means that in fact for each w_i, where $1 \leq i \leq k$, we have that $|w_i|_a \geq 2m$ and $|w_i|_b \geq m + 1$. Since there exists a series of Type 2 transformations that applied to such a word leads to w_1 and another that leads to w_k, we conclude that each of them must be of the form $w_i = a^* v b^*$ with $m = |v|_{ba}$ and $|v|_a = 2m$ and $|v|_b = m + 1$.

This conclude the proof for our second direction. □

The following example shows how the previous result can be used to construct a class of Lyndon conjugates that share the same Parikh matrix.

Example 41. *Following Proposition 40, consider $n = 3$ and look for all possible words with this property. We begin by finding all minimal length binary words that contain 3 subwords ba. These are baaa, baba and bbba. Next add a's to the front and b's to the end of each word, respectively, so that we have a total of 6 a's and 4 b's per word: aaabaaabbb, aaaabababb, and aaaaabbbab. Finally, any number of a's and b's can be added to the front and end of each word, respectively: $a^*aaabaaabbbb^*$, $a^*aaaabababbb^*$, and $a^*aaaaabbbabb^*$. Hence we know that any word of this form is the Lyndon representative of its conjugacy class and shares a Parikh matrix with the two other words stated above. For example, $\Psi(a^2aaabaaabbbb^3) = \Psi(a^2aaaabababbb^3) = \Psi(a^2aaaaabbbabb^3) = \langle\, {}^{8}\,{}^{53}_{7} \,\rangle$.* ◁

Thus far, we presented sufficient conditions for two amiable words not to be $\mathbb{L}$-distinct. Our main result (Theorem 48) shows that these conditions are in fact also the necessary ones. We first present an important observation that we use in the proofs throughout the rest of this paper. For any word $w = w_1w_2$ with $L(w) = w_2w_1$, we call w_2 the right Lyndon suffix of w.

Remark 42. *Consider a binary word $w \in \Sigma_2^*$ with $\Sigma_2 = \{a, b\}$ such that $w = ubav \neq L(w) = avub$ and there exists at least one factor ab in u and/or v. Then applying a* Type 2 *transformation to w using the occurrence of ba on position $|u| + 1$ and any distinct occurrence of ab in w results in an amiable word with the right Lyndon suffix starting in a position different than $|u| + 2$.*

This is clearly shown by examining w before and after the Type 2 transformation is applied. After applying the Type 2 transformation we have $w' = u'abv'$. There is now a b in the $|u| + 2$th position of w', and since we know that there exist at least two a's in w from the Type 2 transformation, it is not possible for the Lyndon conjugate to begin with a b.

The following tools are used in the proof of the final result. We start with a lemma that tells us that if the Parikh vectors of the right Lyndon suffixes of two amiable words are different, then the lengths of these factors must also be unequal.

Lemma 43. *Consider the words $w = w_1w_2 = xabybaz$ and $v = v_1v_2 = xbayabz$ with $w, v \in \Sigma_2^*$ and $a, b \in \Sigma$, such that $w_1, w_2, v_1, v_2 \neq \varepsilon$ and $w_2w_1 = L(w) \neq L(v) = v_2v_1$. If $\phi(w_2) \neq \phi(v_2)$, then $|w_2| \neq |v_2|$.*

Proof. If $\phi(w_2) \neq \phi(v_2)$, then either $|w_2| \neq |v_2|$, or $|w_2| = |v_2|$ with w_2 and v_2 containing a different number of a's and b's. Next we consider the latter case where $\phi(w_2) \neq \phi(v_2)$ and $|w_2| = |v_2|$.

We know from the statement that only one Type 2 transformation needs to be applied to get from $w = xabybaz$ to $v = xbayabz$. Clearly if w_2 and v_2 start anywhere in x, y or z, we do not meet these conditions (their number of a's is equal). The same is true if w_2 and v_2 begin with the first letter of either of the pairs of letters that were changed in the transformation. Hence the only remaining cases are $w_2 = bybaz$ and $v_2 = ayabz$, or $w_2 = az$ and $v_2 = bz$.

However, since w_2 and v_2 represent the start of the Lyndon conjugates of w and v, respectively, none can start with a b when the word contains a's. Hence neither of these cases is possible, and we conclude our proof. □

Furthermore, if two amiable binary words are not the Lyndon representatives of their conjugacy classes, then their corresponding right Lyndon suffixes start at different positions.

Proposition 44. *Consider a binary word* $w \in \Sigma_2^*$ *with* $w = w_1 w_2$ *and* $L(w) = w_2 w_1 \neq w$. *If* $|\mathcal{A}(w, \Psi)| \geq 2$, *then there exists a word* $u = u_1 u_2 \in \mathcal{A}(w, \Psi)$ *with* $L(u) = u_2 u_1$, *such that* $|u_2| \neq |w_2|$.

Proof. We prove this by contradiction. We assume that the right Lyndon suffix of every word associated to $\Psi(w)$ begins in the same position ℓ within the respective words, and show that by using the letter in position ℓ we can apply a Type 2 transformation to one of these words as to obtain another amiable word whose right Lyndon suffix begins in a different position.

Let $w = xabybaz$ and $v = xbayabz$, where $x, y, z \in \Sigma_2^*$ and $a, b \in \Sigma_2$. We know that v must exist since we assume that w is not unambiguous, and also know that Type 2 transformations fully characterise amiable binary words (see Remark 12). Recall that the starting position in w and v of their respective Lyndon conjugates $L(w)$ and $L(v)$ is ℓ.

First consider the case where $\ell \leq |x|$. Since $w \neq L(w)$, we conclude that there must be at least one b right before the ℓth position. Using the first a in $L(w)$ and the b that immediately precedes it, together with the ab that follows x, we apply Remark 42 and hence find a word that is amiable with w and whose right Lyndon suffix starts in a different position.

Now we move on to consider if $|x| + 3 \leq \ell \leq |x| + 2 + |y|$. We use the a at the ℓth position and the b that immediately precedes it to form our ba. Observe that this b exists and in fact $\ell > |x| + 3$ since otherwise the start of the right Lyndon suffix in v is at position $|x| + 2$, which is a contradiction. We can now use the ab following x in w, together with the ba in position $\ell - 1$ in a Type 2 transformation. By applying Remark 42 we conclude that the right Lyndon suffix of the new word begins in a different position to ℓ, again.

Next we consider when $|x| + |y| + 5 \leq \ell \leq |x| + |y| + 4 + |z|$. We know that there must be at least one b in z before the ℓth position, since only a string of a's before ℓ implies that $L(w)$ begins at the $|x| + |y| + 4$ position, which is outside of z. Hence we use Remark 42 to transform w into a word that is amiable with the original whose right Lyndon suffix cannot begin in the ℓth position.

Finally, we consider if $L(w)$ begins at either the $|x| + 1$ position or the $|w| - |z| - 1$ position

$$x\boldsymbol{ab}y\boldsymbol{ba}z \to xbayabz.$$

Clearly v has b's in the two positions where $L(w)$ could have began, so we do not have a right Lyndon suffix beginning at either of these positions. This concludes our proof. □

In line with the above result we also have the following consequence. Note that according to Proposition 33, if all words in $\mathcal{A}(w, \Psi)$ belong to $C(w)$, then $\Psi(w)$ is not $\mathbb{L}$-distinguishable. Therefore, the main feature of the following result is that for all words both them and their (distinct) respective Lyndon representatives are present in $\mathcal{A}(w, \Psi)$.

Proposition 45. *Consider a binary word $w \in \Sigma_2^*$ with $L(w) \neq w$. If for every $v \in \mathcal{A}(w, \Psi)$ we have $L(v) \in \mathcal{A}(w, \Psi)$, then $\Psi(w)$ is $\mathbb{L}$-distinguishable.*

Proof. Let us take w to be the largest lexicographical word in $\mathcal{A}(w, \Psi)$ with $|w|_{ba} = m$, while $|w|_a = p$ and $|w|_b = s$. In this case we have that $w \in \{b^{p-1}a^{p-1}bab^{s-p}, b^{p-1}a^pb^{s-p+1}, a^{p-m}ba^mb^{s-1}\}$, depending on the relation between p and m, where $s \geq m$. Since in $\mathcal{A}(w, \Psi)$ there exists at least a word which is not the Lyndon representative of its class, we are guaranteed that such a w exists.

If $w = b^{p-1}a^{p-1}bab^{s-p}$, then $L(w) = a^{p-1}bab^s$. However, in this case we have that $|L(w)|_{ba} = 1$ and since $p \leq m$ (this is the current situation we consider as to obtain w), any conjugate of a word amiable to w has at least two occurrences of ba. Therefore we reach a contradiction. If $w = b^{p-1}a^pb^{s-p+1}$, then $L(w) = a^pb^s$, which is in contradiction with the fact that our Parikh matrix is ambiguous (since such words are uniquely described).

Finally, let us consider the case when $w = a^{p-m}ba^mb^{s-1}$, where $p \geq m$. We observe that the only possible Lyndon conjugate of this word is $L(w) = a^mb^{s-1}a^{p-m}b$. This is because we assume that both w and $L(w)$ are part of $\mathcal{A}(w, \Psi)$ and they are distinct. However, for $L(w)$ to be a Lyndon representative it must be that $2m \geq p$, with $s - 1 > 1$ in the case of equality. Furthermore, by inspecting the number of ba subwords we have $m = sp - ms - p + m$. Thus $s(p - m) = p \leq 2m$, and since $s \geq m$, it also follows that $p - m \leq 2$.

If $p = m$, then we have a contradiction as our matrix is unambiguous.

If $p = m + 1$, then $s \in \{m, 2m\}$. Thus we get that $w = aba^mb^{m-1}$ and $L(w) = a^mb^{m-1}ab$, for $s = m$, or, when $s = 2m$, we have $w = aba^mb^{2m-1}$ and $L(w) = a^mb^{2m-1}ab$. Since both w and $L(w)$ are in $\mathcal{A}(w, \Psi)$, they must have the same number m of ba subwords. Hence for the former we get that $m - 1 = m$, which is a contradiction, while for the latter we get that $m = 2m - 1$. Therefore, in this case it must be that $m = 1$ and we reach a contradiction with the fact that $abab = w \neq L(w) = abab$, according to our statement.

If $p = m + 2$, then $s = 2m$. In this case we have $w = a^2ba^mb^{2m-1}$ and $L(w) = a^mb^{2m-1}a^2b$. By inspecting again the number of ba's we have that $m = 4m - 2$. Thus, $m = \frac{2}{3}$ which is impossible since m is an integer. □

Furthermore, we remark that there exist words whose corresponding Parikh matrix is not $\mathbb{L}$-distinguishable, despite the fact that they are neither amiable with only Lyndon conjugates, nor are they amiable with only words in their respective conjugacy classes.

Example 46. *Consider the words $w_1 = bbabbaaa$ and $w_2 = bbbaabaa$ which are only amiable to each other. Furthermore $L(w_1) = aaabbabb$ and $L(w_2) = aabaabbb$, and therefore $\Psi_L(w_1) = \Psi_L(w_2)$. Hence w_1 and w_2 are not $\mathbb{L}$-distinct and their Parikh matrix is not $\mathbb{L}$-distinguishable.*

Next we show that, almost, every conjugate of a word associated to a Parikh matrix describing only Lyndon representatives (except for the ones in the above example) is amiable to a word whose Lyndon conjugate has a different Parikh matrix describing it.

Lemma 47. *Consider the binary word $w \in \Sigma_2^*$ with $w_i = L(w_i)$, for all $w_i \in \mathcal{A}(w, \Psi)$. If $u \in C(w)$ and $u \notin \{bbabbaaa, bbbaabaa\}$, then there exists $v \in \mathcal{A}(u, \Psi)$ such that $L(v) \notin \mathcal{A}(w, \Psi)$.*

Proof. Let $|\mathcal{A}(w, \Psi)| = k$. We start our proof by noting that since all of the words in $\mathcal{A}(w, \Psi)$ are the Lyndon representatives of their conjugacy class we have $\mathcal{A}(w, \Psi) = \mathcal{A}(w, \Psi, \Psi_L) = \mathcal{A}(w, \Psi_L)$. Thus, in other words, we have to show that if $u \in C(w)$ and $u \notin \{bbabbaaa, bbbaabaa\}$, then there exists $v \in \mathcal{A}(u, \Psi)$ such that $L(v) \notin \mathcal{A}(w, \Psi)$.

From Proposition 40, we know that there exists some positive integer n such that all our words are of the form $w_m = a^r v_m b^s$, where $1 \leq m \leq k$ and for $n = |v_m|_{ba}$ we have that $|v_m|_a = 2n$ and $|v_m|_b = n + 1$. Since all words of this form are binary and amiable, Remark 29 tells us that in order to have conjugates of these words that share a Parikh matrix, the suffixes of $w_1, w_2, \ldots, w_k$ that we shift, such that we obtain the aforementioned conjugates, either all have the same Parikh vector, or the number of occurrences of a's and b's in each of them leads to a well-order among these suffixes. Note that if two words have the same Parikh vector, they must also have the same length, while the latter condition imposes a difference of at least 2 in the lengths.

Just as in the proof of Proposition 40, we can look at the words that are Lyndon representatives, and share a Parikh matrix, in lexicographical order. By considering the smallest of them, we note that this has to have the form $a^\ell b^m a b^p$, where $\ell, p > 0$ and $m > 1$ since otherwise we can apply a Type 2 transformation as to get a smaller one which is amiable with the current or we only have one element corresponding the the respective Parikh matrix. In the same way, considering the largest, this has to have the form $a^{\ell-m+1} b a^m b^{p+m-1}$, where $\ell > 2m - 1$.

Let us first consider the case when $m = 2$. Thus we consider $a^{\ell}b^2ab^p$ and $a^{\ell-1}ba^2b^{p+1}$, and observe that these words are only amiable to each other. Furthermore, when $\ell = 3$ and $p = 2$ we have our exceptions $\{bbabbaaa, bbbaabaa\}$. For any other values, all conjugates of the corresponding words are part of a set of amiable words of size at least 3, which can be checked by looking at all possible rotations. For example, $bab^pa^{\ell}b$ is also amiable with $ab^{p+1}a^{\ell-1}ba$ and $bab^{p-1}aba^{\ell-2}ba$, while $a^{\ell-s}b^2ab^pa^s$ is also amiable with $a^{\ell-s-1}babab^{p-1}aba^{s-1}$ and $a^{\ell-s-1}ba^2b^{p+1}a^s$. Thus we conclude in this case that each possible conjugate is amiable with a word whose Lyndon conjugate is not corresponding to our Parikh matrix.

For the cases where $m > 2$, we have $a^{\ell}b^mab^p$ and $a^{\ell-m+1}ba^mb^{p+m-1}$ for the minimal and maximal lexicographical elements in $\mathcal{A}(w, \Psi)$, with each containing $\ell(m+p)+p$ occurrences of ab. Now, consider all of the possible conjugates of elements in $\mathcal{A}(w, \Psi)$ obtained by a shift of no more than $\ell-m+1$ (up to ℓ) symbols a to the end. Then it must be the case that we can apply a Type 2 transformation to the obtained words by using the last occurrences of ab and of ba in the word as to generate a word that it is amiable to ours. However, observe that this word is not the conjugate of any Lyndon representative from $\mathcal{A}(w, \Psi)$, since shifting now the last block of a's in it back to the beginning would render the Lyndon conjugate of this word which has as many as $\ell(m + p) + p + 1$ occurrences of ba.

If on the other hand the shift is of at most p (up to $p+m-1$) symbols b from the back to the front, we can consider the first occurrences of ba and ab in the word to apply a Type 2 transformation and follow the exact previously described strategy as to reach a contradiction.

In the rest of the cases, when a factor ab is shifted from the beginning to the end, we can consider this factor in the respective conjugate together with the first occurrence of ba preceding it (such an occurrence exists, in the worst case right at the end of the last block of b's in the Lyndon representative). After we apply a Type 2 transformation to this word we get, according to Remark 42, a new word that is amiable to ours but whose right Lyndon suffix begins in a different position and is therefore $\mathbb{L}$-distinct. Since also the starting position of the right Lyndon suffix differs by exactly 1, it is also the case that this new word cannot be obtained from a word in $\mathcal{A}(w, \Psi)$ as the number of a's and b's that would presumably be shifted is not strictly increasing, or decreasing, as compared to how our conjugate was obtained.

This concludes our proof. □

We end this section by giving our main result that characterises all binary words whose Parikh matrix is not $\mathbb{L}$-distinguishable.

Theorem 48. *For the binary alphabet, a Parikh matrix is not* $\mathbb{L}$*-distinguishable if and only if there exists a word* $w \in \Sigma_2^*$ *associated with that matrix such that* w *meets at least one of the following criteria:*

- $w \in \{aabb, ababbb, aababb, aabbab, aaabab, bbabbaaa, bbbaabaa\}$
- $w = a^*vb^*$ *and for* $n = |v|_{ba}$ *we have that* $|v|_a = 2n$ *and* $|v|_b = n + 1$

Proof. The 'if' direction was proven earlier in the paper when Propositions 33, 37, 38, 40 and Example 46, describing these situations, were introduced. Therefore we move on to consider the 'only if' direction.

Let us now consider an ambiguous Parikh matrix (there are at least two words describing it). If the matrix is not $\mathbb{L}$-distinguishable it must be that the Lyndon representatives of all of the words associated to the Parikh matrix share in their turn a Parikh matrix as well. Namely, a Parikh matrix is not $\mathbb{L}$-distinguishable if all of the words it describes are either:

1. members of the same conjugacy class, from Proposition 33,
2. Lyndon representatives, from Proposition 38,
3. conjugates of Lyndon representatives that share a Parikh matrix.

We know from Proposition 37 that the only words that are amiable with only other members of their own conjugacy class are those in the set $\{aabb, ababbb, aababb, aabbab, aaabab\}$.

The case when the Parikh matrix is in fact one corresponding to only Lyndon representatives is precisely the one discussed in Propositions 38 and 40 and covered by our statement.

If, on the other hand, there exist two amiable words v and w, such that $v = L(v)$ while $w \neq L(w)$, then according to Proposition 45 the matrix that they describe is $\mathbb{L}$-distinguishable.

Finally, let us assume that our Parikh matrix contains no Lyndon representatives but at least two words from different conjugacy classes. We have that either there exists an $\mathbb{L}$-Parikh matrix corresponding uniquely to our words, or there exist other further words associated to the $\mathbb{L}$-Parikh matrix. For the former case, following Lemma 47, except for *bbabbaaa* and *bbbaabaa* there exist no words that are both amiable and whose Lyndon representatives uniquely describe the same Parikh matrix.

For the latter case we can again consider the existence of a smallest lexicographical word $v = a^{p-1}b^m ab^{s-m}$ and of a largest one $w \in \{a^{p-m}ba^m b^{s-1}, b^{p-1}a^{p-1}bab^{s-p}, b^{p-1}a^p b^{s-p+1}\}$, depending on the relation between p and m. These must exist and according to our hypothesis they are not the Lyndon representatives of their respective classes. We easily deduce that in this case $p = 2$, $m \leq s \leq 2m$, and analysing all possible combinations of v and w get contradictions in all cases. Namely, we get that either $v = abab$ or aba, which are not possible descriptions of such words.

Hence the criteria outlined in the two points in our statement are the only criteria that result in a Parikh matrix not being $\mathbb{L}$-distinguishable. □

5. Conclusion and Future Work

In this paper, we have shown that, when used in addition to Parikh matrices, $\mathbb{P}$-Parikh and $\mathbb{L}$-Parikh matrices reduce the ambiguity of a word in most cases. From Corollary 19, we learn that $\mathbb{P}$-Parikh matrices, despite not being able to reduce the ambiguity of a Parikh matrix that describes words in a binary alphabet, shine when it comes to reducing the ambiguity of words in larger alphabets (Proposition 17). On the other hand, we find that $\mathbb{L}$-Parikh matrices reduce the ambiguity of most binary words, with the few exceptions from Theorem 48, which have all been shown to be rare occurrences within the case of the binary alphabet. Thus, using both tools together leads to a reduction in ambiguity in most cases.

Before looking into potential future work, let us consider the following examples.

Example 49. *Consider the word $w = a^4bab^{16}a^3ba^3b$ with $|w|_a = 11$, $|w|_b = 19$, and $|w|_{ab} = 103$. It is easy to see that $w = L(w)$. Since $|w|_{ba} = 106$, we note that the number of occurrences of ab is not necessarily larger than that of the occurrences of ba. Furthermore, shifting the factor a^3b to the beginning of w gives us a word which has 149 occurrences of ab.* ◁

The above example shows that the Lyndon representative of a conjugacy class does not necessarily contain the highest number of ab subwords in the class, nor is its number of ab subwords higher than the number of occurrences of ba.

Example 50. *Consider the word $v = a^6b^4ab^3$. This word is the Lyndon representative of its conjugacy class, and apart from $w = a^3ba^4b^6$ the rest of the words associated to $\Psi(v)$ are from*

$$S_1 = \{a^6b^4ab^3, a^5bab^2ab^4, a^5b^2a^2b^5, a^4ba^2bab^5\}.$$

As one can see, except for w, all the words amiable to v are in fact the Lyndon representatives of their conjugacy classes.

Let us also define the set of words that is obtained by removing the final two b's from the end of each word in S_1.

$$S_2 = \{a^6b^4ab, a^5bab^2ab^2, a^5b^2a^2b^3, a^4ba^2bab^3\}.$$

Now, let us consider the Lyndon conjugate $L(w) = a^4b^6a^3b$. This word is in fact amiable to a multitude of words (more than the ones in S_1). However, what is to observe is that among the words amiable to $L(w)$ there are also all of the words v_2v_1, where $v_2 = bb$, while v_1 can be replaced with each of the elements in the set S_2. ◁

The above examples show that a situation where there exist words u and w such that u is amiable to $L(w)$, while w is amiable to $L(u)$, is possible. Furthermore, they also strengthen our definition of $\mathbb{L}$-distinguishability for Parikh matrixes. That is because, by looking at Proposition 45, if we were to have for $\mathbb{L}$-distinguishability only a reduction in the number of described words, for one of the conjugates, then we would have encountered further issues. Nevertheless, we think that this research direction deserves further consideration.

Going forward, we wish to characterise words that are described uniquely by both types of matrices, respectively, as well as quantifying the ambiguity reduction permitted by both notions. Theorem 48 tells us that there are very few binary words whose Parikh matrix ambiguity cannot be reduced by $\mathbb{L}$-Parikh matrices. Future research on $\mathbb{L}$-Parikh matrices could also include an analysis similar to the one done in Proposition 17.

Finally we present a conjecture on the types of words that might be described by a Parikh matrix that is $\mathbb{P}$-distinguishable. We know that the presence of a certain type of factor, described in Proposition 16, in a word means that its Parikh matrix is $\mathbb{P}$-distinguishable. This conjecture implies that the presence of this factor is the *only* way that the ambiguity of a word could be reduced by $\mathbb{P}$-Parikh matrices.

Conjecture 51. *For any word $w \in \Sigma_n^*$, if $\Psi(w)$ is $\mathbb{P}$-distinguishable, then there exists a word amiable with w which contains a factor a_ia_j, where $|i - j| > 1$.*

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