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Robert Mercas · Daniel Reidenbach (Eds.)

# Combinatorics on Words

12th International Conference, WORDS 2019  
Loughborough, UK, September 9–13, 2019  
Proceedings

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# Preface

This volume of *Lecture Notes in Computer Science* contains the proceedings of the 12th International Conference WORDS 2019, which was organized by the Department of Computer Science at Loughborough University and held during September 9–13, 2019, in Loughborough, UK.

WORDS is the main conference series devoted to the mathematical theory of words. In particular, the combinatorial, algebraic and algorithmic aspects of words are emphasized. Motivations may also come from other domains such as theoretical computer science, bioinformatics, digital geometry, symbolic dynamics, numeration systems, text processing, number theory, etc.

The conference WORDS takes place every two years. The first conference of the series was held in Rouen, France in 1997. Since then, the locations of WORDS conferences have been: Rouen, France (1999), Palermo, Italy (2001), Turku, Finland (2003 and 2013), Montréal, Canada (2005 and 2017), Marseille, France (2007), Salerno, Italy (2009), Prague, Czech Republic (2011), and Kiel, Germany (2015).

For the fourth time in the history of WORDS, a refereed proceedings volume was published in the *Lecture Notes in Computer Science* series of Springer. There were 34 submissions, from 17 countries, and each of them was reviewed by three or four referees. The selection process was undertaken by the Program Committee with the help of generous reviewers. From these submissions, 21 papers were selected to be published and presented at WORDS.

In addition to the contributed presentations, WORDS featured a session *In Memoriam: Aldo de Luca*, as well as six invited talks:

- Florin Manea (Kiel University, Germany): “Matching Patterns with Variables”
- Svetlana Puzynina (Saint Petersburg State University, Russia): “Abelian Properties of Words”
- Antonio Restivo (University of Palermo, Italy): “On Sets of Words of Rank Two”
- Gwenaël Richomme (Université Paul-Valéry Montpellier 3, France): “ $S$ -adicity and Property Preserving Morphisms”
- Aleksu Saarela (University of Turku, Finland): “Independent Systems of Word Equations: From Ehrenfeucht to Eighteen”
- Kristina Vušković (University of Leeds, UK): “Structure of Graph Classes and Algorithms”

The present volume also includes the papers of five of these invited talks.

We take this opportunity to warmly thank all the invited speakers and all the authors for their contributions. We are also grateful to all Program Committee members and the additional reviewers for their hard work that led to the selection of papers published in this volume. The reviewing process was facilitated by the EasyChair conference system, created by Andrej Voronkov. Special thanks are due to Alfred Hofmann and Anna Kramer and the Lecture Notes in Computer Science team at Springer for having

granted us the opportunity to publish this special issue devoted to WORDS 2019 and for their help during the process. We are also grateful for the support received from the School of Science, and especially the administrative team at Loughborough University, the Institute of Advanced Studies at Loughborough University, the European Association for Theoretical Computer Science, and the journal *Algorithms*. We gratefully acknowledge the contribution by Alex Hewitson and Luke Bellamy, who created the conference website, and Ovidiu Toader (from Brainik), who designed the conference poster. Finally, we are much obliged to a number of people who contributed to the success of the conference: our colleagues Joel Day and Manfred Kufleitner, and our students Laura Hutchinson and Alex Smith. Our warmest thanks for their assistance in the organization of the event!

July 2019

Robert Mercas  
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# On Families of Limit $S$ -adic Words (Invited Talk)

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**Abstract.** Given a set  $S$  of morphisms, an infinite word is limit  $S$ -adic if it can be recursively desubstituted using morphisms in  $S$ . Substitutive-adicity arises naturally in various studies especially in studies on infinite words with factor complexity bounded by an affine function. In the literature, when a family  $F$  of infinite words defined by a combinatorial property  $P$  appears to be  $S$ -adic for some set  $S$  of morphisms, it is very rare that the whole set of limit  $S$ -adic words coincides with  $F$ . The aim of the talk is to survey such situations in which necessarily morphisms of  $S$  preserve the property  $P$  of infinite words.

**Keywords:**  $S$ -adicity · Property preserving morphisms

We assume that readers are familiar with combinatorics on words; for omitted definitions see, e.g., [5, 11, 12]. All the infinite words considered in this abstract are right infinite words.

As explained with more details in [3], the terminology  $S$ -adic was introduced by S. Ferenczi [8]. Without context, letter  $S$  refers to term “substitution” (and we will sometimes use the terminology *substitutive-adicity* instead) and in more precise definitions, it refers to a set  $S$  of (nonerasing) morphisms. An infinite word  $\mathbf{w}$  is said  $S$ -adic if there exist a sequence  $(f_n)_{n \geq 1}$  of morphisms in  $S$  and a sequence of letters  $(a_n)_{n \geq 1}$  such that  $\lim_{n \rightarrow +\infty} |f_1 f_2 \cdots f_n(a_{n+1})| = +\infty$  and  $\mathbf{w} = \lim_{n \rightarrow +\infty} f_1 f_2 \cdots f_n(a_{n+1}^\omega)$ . The sequence  $(f_n)_{n \geq 1}$  is called a *directive word* of  $\mathbf{w}$ . Assume limits  $\mathbf{w}_k = \lim_{n \rightarrow +\infty} f_k f_{k+1} \cdots f_n(a_{n+1}^\omega)$  exist for all  $k$ . Observe that  $\mathbf{w}_1 = \mathbf{w}$  and  $\mathbf{w}_n = f_n(\mathbf{w}_{n+1})$  for all  $n \geq 1$ , that is,  $\mathbf{w}$  can be infinitely desubstituted using morphisms in  $S$ . Following [1] (where is used terminology *limit point* of a sequence of substitutions), we say that  $\mathbf{w}$  is a *limit  $S$ -adic word* when such sequences  $(f_n)_{n \geq 1}$  and  $(\mathbf{w}_n)_{n \geq 1}$  exist. Observe that the set of limit  $S$ -adic words is the minimal set of infinite words  $X$  such that  $X = \bigcup_{f \in S} f(X)$  that we could denote by abuse of notation  $X = S(X)$  to emphasize the fact that limit  $S$ -adic words are generalizations of fixed points of morphisms (and even of morphic sequences).

Limit substitutive-adicity arises naturally in various studies as, for instance, this of Sturmian words (see, e.g. [12, Chap. 2] and [13, Chap. 5]). It is well-known (and easy to prove) that any Sturmian word is a limit  $S_{sturm}$ -adic word with  $S_{sturm} = S_a \cup S_b$ ,  $S_a = \{L_a, R_a\}$ ,  $S_b = \{L_b, R_b\}$ ,  $L_a(a) = a = R_a(a)$ ,  $L_a(b) = ab$ ,  $R_a(b) = ba$ ,

$L_b(a) = ba$ ,  $R_b(a) = ab$ ,  $L_b(b) = b = R_b(b)$ . But not all limit  $S_{sturm}$ -adic words are Sturmian. Only the limit  $S_{sturm}$ -adic words whose directive word contains infinitely many elements of  $S_a$  and infinitely many elements of  $S_b$  are Sturmian. This condition can be described using infinite paths with prohibited segments in an automaton (or a graph), here with two states (or vertices), one for each set  $S_a$  and  $S_b$ . This kind of condition with an automaton to characterize allowed directive words is also used, for instance, in the characterization of words for which the first difference of factor complexity is bounded by 2 [10] or in the characterization of sequences arising from the study of multidimensional continued fraction algorithm (see for instance [4, 6]).

In [14], answering a question of G. Fici, the author characterizes in term of limit  $S$ -adicity the family of so-called LSP infinite words, that is the words having all their left special factors as prefixes. For this he determines a suitable set  $S_{bLSP}$  of morphisms and an automaton recognizing allowed infinite desubstitutions. As the obtained characterization is quite evolved, a second part of [14] considers the question of finding a simpler  $S$ -adic characterization. It is proved that, unfortunately, it does not exist any set of morphisms  $S$  such that the family of LSP infinite words is (exactly) the family of limit  $S$ -adic words except in the binary case.

The aim of the talk is to consider the question: which are the known families of infinite words defined by a combinatorial property  $P$  that correspond to a family of limit  $S$ -adic words for some set of  $S$  of morphisms? In [14], it was observed that when such a situation arises necessarily morphisms of  $S$  preserve the property  $P$  of infinite words. In what follows we will say that a morphism preserves words of a family  $F$  if it maps any word on this family on another word of this family.

In the case of Sturmian words, morphisms in  $S_{sturm}$  indeed preserve Sturmian words. That all  $S_{sturm}$ -adic infinite words are not Sturmian comes from the fact that some limit  $S_{sturm}$ -adic words are periodic. At this stage, readers should remember that Sturmian words are the aperiodic binary balanced words (where balanced means that the numbers of  $a$  occurring in any factors  $u$  and  $v$  of equal length may differ by at most one). Morphisms in  $S_{sturm}$  preserve binary balanced words and consequently limit  $S_{sturm}$ -adic words are the infinite binary balanced words.

Actually it is easy to observe that the family of Sturmian words corresponds to a family of limit substitutive-adic words, more precisely to the family of limit  $S'_{sturm}$ -adic words where  $S'_{sturm} = \bigcup_{n \geq 1} S'_a S'_b \cup S'_b S'_a$ . Morphisms of  $S'_{sturm}$  are obtained composing morphisms to enforce infinite occurrences of morphisms of each of the sets  $S_a$  and  $S_b$  in directive words of Sturmian words. One can note that in this case the set of morphisms  $S'_{sturm}$  is infinite.

A similar situation holds for the Arnoux-Rauzy words [2]. They are limit- $S_E$ -adic words for a finite set of morphisms that generalize Sturmian words and they correspond to a family of limit substitutive-adic words for an infinite set of morphisms (obtained concatenating morphisms in  $S_E$ ). The morphisms in  $S_E^*$  are morphisms that preserve episturmian words [7, 9] and the set of episturmian words is exactly the set of  $S_E$ -adic words.

Although limit  $S$ -adicity reveals itself to be a useful tool to study some combinatorial properties, although the notion of property preserving morphisms is often considered to generate words with interesting properties, the family of right infinite

balanced words and the family of episturmian words seems to be the unique known families of words that correspond exactly to a family of limit  $S$ -adic words with  $S$  a finite set of morphisms.

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