The Three-Squares Lemma for Partial Words with One Hole*

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March 16, 2011

Abstract

Partial words, or sequences over a finite alphabet that may have do not know symbols or holes, have been recently the subject of much investigation. Several interesting combinatorial properties have been studied such as the periodic behavior and the counting of distinct squares in partial words. In this paper, we extend the three-squares lemma on words to partial words with one hole. This result provides special information about the squares in a partial word with at most one hole, and puts restrictions on the positions at which periodic factors may occur, which is in contrast with the well known periodicity lemma of Fine and Wilf.

Keywords: Combinatorics on words; Partial words; Squares; Fine and Wilf's periodicity lemma; Three-squares lemma.

1 Introduction

A *square* in a full word w has the form uu for some factor u of w. A well known problem is the determination of $\sigma(n)$, the maximum number of distinct squares in any full word of length n, where experiment strongly suggests that $\sigma(n) < n$. With this problem progress has been made: Fraenkel and Simpson showed that $\sigma(n) \leq 2n - 2$ [15], a result recently proved somewhat more simply by Ilie [17], then later improved to $\sigma(n) \leq 2n - \Theta(\log n)$ [18].

*This material is based upon work supported by the National Science Foundation under Grant No. DMS–0754154. The Department of Defense is also gratefully acknowledged.

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In fact, it was shown that at each position there are at most two distinct squares whose last occurrence starts.

In order to show that $\sigma(n) < n$, we need to somehow limit to less than one the average number of squares that begin at the positions of w. This requirement draws attention to positions i where two or more squares begin. Is it true that at positions "neighbouring" to i, no squares can begin? Perhaps the most famous theoretical result restricting periodicity is Fine and Wilf's "periodicity lemma" stated as Theorem 1.

Theorem 1 ([14]). *If a full word w has two periods p, q and $|w| \geq p + q - \gcd(p, q)$, then w has also a period $\gcd(p, q)$.*

Unfortunately this theorem provides no special information about the squares, and it puts no restrictions on the positions at which periodic factors may occur. A result that provides such information is the following "three-squares lemma" stated as Theorem 2.

Theorem 2 ([12, 19]). *If w^2, v^2, and u^2 are three squares starting at the same position (not necessarily last occurrences) in a full word, such that $v \notin w^*$, $|w| < |v| < |u|$ and w is primitive, then $|w| + |v| \leq |u|$.*

The main result in [13] is essentially a generalization of this result that allows v to be offset by k positions from the start of u^2, and that does not always require complete squares u^2 and v^2, only sufficiently long factors of periods $|u|$ and $|v|$. Moreover, as a corollary, it specifies exactly the periodic behavior in the word. It is hoped that, with the help of such results, it will be possible to establish, or at least make progress with, the conjecture that $\sigma(n) < n$.

The counting of distinct squares in partial words was recently initiated and revealed surprising results [8, 9]. In this case, a square in a partial word over a given alphabet has the form uu' where u' is *compatible* with u, and consequently, such square is compatible with a number of full words over the alphabet that are squares. In [9], it was shown that for partial words with one hole, there may be more than two squares that have their last occurrence starting at the same position. There it was proved that if such is the case, then the hole is in the shortest square. Furthermore, it turned out that the length of the shortest square is at most half the length of the third shortest square [8]. As a result, it was shown that the number of distinct full squares compatible with factors of a partial word with one hole of length n is bounded by $\frac{7n}{2}$.

Although Fine and Wilf's Theorem 1 has extensively been studied in the context of partial words [1, 2, 4, 7, 11, 16, 20, 21], such is not the case of the Three-Squares Theorem 2. In this paper, we prove the three-squares theorem in the context of partial words with one hole.

2 Preliminaries

Fixing a nonempty finite set of letters or an *alphabet* A, a *partial word* u of length $|u| = n$ over A is a partial function $u : \{0, \ldots, n-1\} \to A$. For $0 \leq i < n$, if $u(i)$ is defined, then i belongs to the *domain* of u, denoted by $i \in D(u)$, otherwise i belongs to the *set of holes* of u, denoted by $i \in H(u)$ (a partial word u such that $H(u) = \emptyset$ is also called a *full word*). The unique word of length 0, denoted by ε, is called the *empty* word. For convenience, we will refer to a partial word over A as a word over the enlarged alphabet $A_\diamond = A \cup \{\diamond\}$, where $\diamond \notin A$ represents a hole. For partial words u, v, and w, if $w = uv$, then u is a *prefix* of w, denoted by $u \leq w$, and if $v \neq \varepsilon$, then u is a *proper prefix* of w, denoted by $u < w$. If $w = xuy$, then u is a *factor* of w. The set of all words (respectively, nonempty words, partial words, nonempty partial words) over A of finite length is denoted by A^* (respectively, A^+, $A_\diamond^*$, $A_\diamond^+$).

2.1 Periodicity

A *strong period* of a partial word u is a positive integer p such that $u(i) = u(j)$ whenever $i, j \in D(u)$ and $i \equiv j \bmod p$. In this case, we call u *strongly p-periodic*. A *weak period* of u is a positive integer p such that $u(i) = u(i+p)$ whenever $i, i+p \in D(u)$. In this case, we call u *weakly p-periodic*. Note that every weakly p-periodic full word is strongly p-periodic but this is not necessarily true for partial words.

Fundamental results on periodicity of full words include the theorem of Fine and Wilf, which considers the simultaneous occurrence of different periods in a word. The following theorem extends this result to partial words with one hole.

Theorem 3 ([1])**.** *Let $w \in A_\diamond^*$ be weakly p-periodic and weakly q-periodic. If $H(w)$ is a singleton and $|w| \geq p + q$, then w is strongly $\gcd(p, q)$-periodic.*

2.2 Containment and compatibility

The partial word u is *contained in* the partial word v, denoted by $u \subset v$, provided that $|u| = |v|$, all elements in $D(u)$ are in $D(v)$, and for all $i \in D(u)$ we have that $u(i) = v(i)$. For example, $a\diamond b\diamond bba \subset aab\diamond bba$. The *greatest* *lower bound* of a pair of partial words u and v of equal length is the partial word $u \wedge v$ such that $(u \wedge v) \subset u$ and $(u \wedge v) \subset v$, and for all partial words w which satisfy $w \subset u$ and $w \subset v$ we have that $w \subset (u \wedge v)$.

A partial word u is *primitive* if there exists no word v such that $u \subset v^n$ with $n \geq 2$. If u is a nonempty partial word, then there exist a primitive word v and a positive integer n such that $u \subset v^n$. Uniqueness holds for full words but not for partial words as seen with $u = \diamond a$ where $u \subset a^2$ and $u \subset ba$ for distinct letters a and b. Note that if $u \wedge v$ is primitive for some partial words u and v of equal length, then both u and v are primitive.

The partial words u and v are *compatible*, denoted by $u \uparrow v$, provided that there exists w such that $u \subset w$ and $v \subset w$. An equivalent formulation of compatibility is that $|u| = |v|$ and for all $i \in D(u) \cap D(v)$ we have that $u(i) = v(i)$. The following rules are useful for computing with partial words: (1) *Multiplication:* If $u \uparrow v$ and $x \uparrow y$, then $ux \uparrow vy$; (2) *Simplification:* If $ux \uparrow vy$ and $|u| = |v|$, then $u \uparrow v$ and $x \uparrow y$; and (3) *Weakening:* If $u \uparrow v$ and $w \subset u$, then $w \uparrow v$.

The following lemmas will be useful for our purposes.

Lemma 1 ([1])**.** *Let $x, y \in A^+$ and let $z \in A_\diamond^*$ be such that $H(z)$ is a singleton. If $z \subset xy$ and $z \subset yx$, then $xy = yx$, that is, x and y are powers of a common word.*

Lemma 2 ([5])**.** *Let $x, y, z \in A_\diamond^*$ be such that $|x| = |y| > 0$. Then $xz \uparrow zy$ if and only if xzy is weakly $|x|$-periodic.*

Lemma 3 ([6])**.** *Let $x, y \in A_\diamond^+$ and $z \in A^*$. If $xz \uparrow zy$, then there exist $v, w \in A^*$ and an integer $n \geq 0$ such that $x \subset vw$, $y \subset wv$, and $z = (vw)^n v$. Consequently, if $xz \uparrow zy$, then xzy is strongly $|x|$-periodic.*

If $u = u_1 u_2$ for some nonempty compatible partial words u_1 and u_2, then u is called a *square*. Whenever we refer to a square $u_1 u_2$ it will imply that $u_1 \uparrow u_2$.

3 The three-squares theorem

The main result in this paper is a generalization of the three-squares theorem to partial words with one hole.

Let ww', vv', and uu' be three squares starting at the same position (not necessarily last occurrences) in a partial word with one hole, such that $|w| < |v| < |u|$ and $(w \wedge w')$ is primitive.

- In Section 3, we will prove that $|u| \geq 2|w|$ (see Theorem 4).

- In Section 4, we will prove that if $|v| \geq 2|w|$, then $|w| + |v| \leq |u|$ (see Theorem 5).
- In Section 5, we will prove that if $|v| < 2|w|$ and the hole is not in ww', then $|w| + |v| \leq |u|$ (see Theorem 6). We will also show that we cannot completely get rid of the assumption that "the hole is not in ww'."
- In Section 6, we will describe precisely when we can guarantee that $|w| + |v| \leq |u|$ under the conditions that $|v| < 2|w|$ and the hole is in ww' (see Theorem 7).
- In Section 7, we will state our main theorem (see Theorem 8), and will conclude with some remarks.

We start with a lemma that extends synchronization to partial words with one hole (synchronization is the property that a full word w is primitive if and only if in ww there exist exactly two factors equal to w, namely the prefix and the suffix).

Lemma 4 ([3]). *Let w be a partial word with at most one hole. Then w is primitive if and only if $ww \uparrow xwy$ for some x and y implies $x = \varepsilon$ or $y = \varepsilon$.*

Using the previous lemma, we can easily prove the following.

Lemma 5. *If ww' is a square with one hole such that $ww' \uparrow xw''y$ for some nonempty partial words x and y, and some partial word w'' satisfying $(w \wedge w') \subset w''$, then $(w \wedge w')$ is not primitive.*

Proof. For the sake of contradiction, suppose that $(w \wedge w')$ is primitive. By weakening, we get that $(w \wedge w')(w \wedge w') \uparrow x(w \wedge w')y$. By lemma 4, $x = \varepsilon$ or $y = \varepsilon$, a contradiction. □

We now prove that $|u| \geq 2|w|$.

Theorem 4. *Let ww', vv', and uu' be three squares starting at the same position (not necessarily last occurrences) in a partial word with one hole, such that $|w| < |v| < |u|$ and $(w \wedge w')$ is primitive. Then $|u| \geq 2|w|$.*

Proof. Since $|w| < |v| < |u|$, let us denote $v = wz_1$ and $u = vz_2$, for some partial words z_1 and z_2. For the sake of contradiction, we suppose that $|u| < 2|w|$. Denote $ww' = uz_3$, for some partial word z_3. We have $w' = z_1z_2z_3$, $w = z'_1z'_2z'_3$, $v = z'_1z'_2z'_3z_1$, and $u = z'_1z'_2z'_3z_1z_2$, where $z'_i \uparrow z_i$ for all $i \in \{1, 2, 3\}$. Since $v \uparrow v'$, we get that there exists a partial word z_4 such

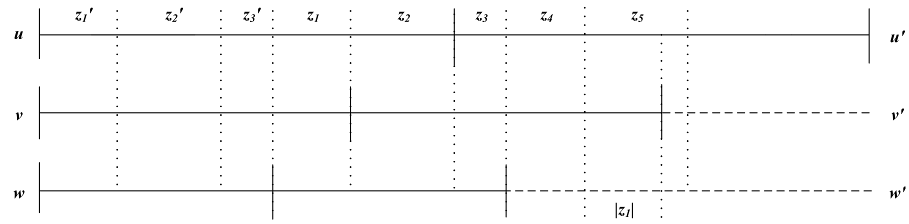

Figure 1: The case when $|w| < |v| < |u| < 2|w|$

that $z_2z_3z_4$ is a prefix of v' and $|z_4| = |z_1|$, and by looking at the prefixes of length $|w|$ of u and u', we get that there exists a partial word z_5, with $|z_5| = |z_2|$, such that $z_1'z_2'z_3' \uparrow z_3z_4z_5$ (see Figure 1).

First, assume that the hole is not in ww'. Here $w = w'$, and consequently $z_1 = z_1', z_2 = z_2'$, and $z_3 = z_3'$. Since $v \uparrow v'$, $z_1z_2z_3 \uparrow z_2z_3z_4$, and since $u \uparrow u'$, $z_1z_2z_3 \uparrow z_3z_4z_5$. The former implies $z_1z_2z_3z_4$ is weakly $|z_1|$-periodic by Lemma 2, while the latter implies that $z_1z_2z_3z_4z_5$ is weakly $|z_1z_2|$-periodic, and hence, so are $z_1z_2z_3z_4$ and $z_2z_3z_4z_5$. By Theorem 3, $z_1z_2z_3z_4$ is strongly $\gcd(|z_1|, |z_1z_2|)$-periodic. Let x be the prefix of z_1 of length $\gcd(|z_1|, |z_1z_2|)$, and so $z_1 = x^m$ and $z_1z_2 = x^{m+n}$ for some integers $m, n > 0$. Furthermore, since $z_2z_3z_4 \subset z_1z_2z_3$ and $z_3z_4z_5 \subset z_1z_2z_3$, we get that $z_2z_3z_4 \uparrow z_3z_4z_5$, and so $z_2z_3z_4z_5$ is weakly $|z_2|$-periodic by Lemma 2. Hence, by Theorem 3, $z_2z_3z_4z_5$ is strongly $|x| = \gcd(|z_2|, |z_1z_2|)$-periodic. It must be the case that $z_3 = (x'x'')^px'$, $z_4 \subset (x''x')^m$, and $z_5 \subset (x''x')^n$, for a factorization $x'x''$ of x and nonnegative integers p, m, and n. Let z_5' be the prefix of length $|x|$ of z_5. Since $v \uparrow v'$, we get by simplification, $z_1z_2z_3x \uparrow z_2z_3z_4z_5'$. Using simplification again, we get that $x \uparrow z_5'$, and so $z_5' \subset x'x''$. The latter and the fact that $z_5' \subset x''x'$ give us that x' and x'' are powers of the same word by Lemma 1, which leads to a contradiction with our initial assumption that $w = z_1z_2z_3$ is primitive.

Now, assume that the hole is in ww'. There are six cases to consider: Case 1 (the hole is in z_3), Case 2 (the hole is in z_2), Case 3 (the hole is in z_1), Case 4 (the hole is in z_3'), Case 5 (the hole is in z_2'), and Case 6 (the hole is in z_1'). The arguments are similar to those found in [8], and so we will only treat here Cases 2 and 5.

Case 2. The hole is in z_2.

Hence, $w' = z_1z_2z_3$ and $w = z_1z_2'z_3$, where $z_2 \subset z_2'$, $v = z_1z_2'z_3z_1$, and $u = z_1z_2'z_3z_1z_2$. Since $z_1z_2'z_3$ and $z_2z_3z_4$ are prefixes of v and v', respectively, and $|z_1z_2'z_3| = |z_2z_3z_4|$, we get $z_1z_2'z_3 \uparrow z_2z_3z_4$ and $z_1z_2z_3 \uparrow z_2z_3z_4$ by applying weakening. Using Lemma 2, we get

$$z_1z_2z_3z_4 \text{ is weakly } |z_1|\text{-periodic} \tag{1}$$

Now, since $w' \subset w$, by looking at the prefixes of v' and u' of length $|w|$, we can observe that $z_2z_3z_4 \subset w$ and $z_3z_4z_5 \subset w$, respectively. From these, we get $z_2z_3z_4 \uparrow z_3z_4z_5$. Using Lemma 3, we get

$$z_2z_3z_4z_5 \text{ is strongly } |z_2|\text{-periodic} \tag{2}$$

Finally, by comparing the prefixes of length $|w|$ of u and u', we have $z_1z_2'z_3 = z_3z_4z_5$. Using Lemma 3, it results that

$$z_1z_2'z_3z_4z_5 \text{ is strongly } |z_1z_2|\text{-periodic} \tag{3}$$

From (1) and (3) we get that $z_1z_2z_3z_4$ is weakly $|z_1|$- and $|z_1z_2|$-periodic. Applying Theorem 3, we have that $z_1z_2z_3z_4$ is strongly $\gcd(|z_1|, |z_1z_2|)$-periodic. Hence, there exists a full word x of length $\gcd(|z_1|, |z_1z_2|)$, such that $z_1 = x^m$ and $z_1z_2 \subset x^{m+n}$, for some integers $m, n > 0$. From (2) and (3) we get that $z_2z_3z_4z_5$ is strongly $|z_2|$- and $|z_1z_2|$-periodic. Applying Theorem 3, $z_2z_3z_4z_5$ is strongly $\gcd(|z_2|, |z_1z_2|)$-periodic. It follows that $z_1z_2z_3z_4z_5$ is strongly $|x|$-periodic.

Because z_1 and z_5 share a prefix of length $\min(|z_1|, |z_5|)$, and $|z_5| = |x^n|$, we get that $z_5 = x^n$. Since $z_3z_4z_5$ is strongly $|x|$-periodic, $|z_5| \geq |x|$ and $|z_4| = |x^m|$, we get that $z_4 = x^m = z_1$. Since $z_1z_2z_3$ is strongly $|x|$-periodic, it results that $z_3 = (x'x'')^p x'$ for some words x', x'' such that $x = x'x''$ and some integer $p \geq 0$. By looking at the prefixes of length $|w|$ of u and u', we notice that $z_1z_2'z_3 = z_3z_1z_5$. This implies that $x'x'' = x''x'$. Thus there exist integers q and r with $q, r \geq 0$ and a word y such that $x' = y^q$ and $x'' = y^r$. But since $w' = z_1z_2z_3$, we get, again, a contradiction with the assumption that w' is primitive.

Case 5. The hole is in z_2'.

Looking at the prefixes of length $|w|$ of v and v', we have $z_1z_2'z_3 \uparrow z_2z_3z_4$. Applying weakening and Lemma 2, we get that $z_1z_2'z_3z_4$ is weakly $|z_1|$-periodic. Also, by looking at the prefixes of length $|w|$ of u and u' we get that $z_1z_2'z_3 \uparrow z_3z_4z_5$. By applying Lemma 3, we get that $z_1z_2'z_3z_4z_5$ is strongly $|z_1z_2|$-periodic. Using Theorem 3, it follows that $z_1z_2'z_3z_4$ is strongly $\gcd(|z_1|, |z_1z_2|)$-periodic. Hence, there exists x such that $z_1 = x^m$ and $z_1z_2' \subset x^{m+n}$, for some positive integers m, n and $|x| = \gcd(|z_1|, |z_1z_2|)$. Hence, we have $z_3 = (x'x'')^p x'$ and $z_4 = (x''x')^m$, where $x = x'x''$ and $p \geq 0$.

Since the hole is in z_2', either there are integers n_1, n_2 and word x_1' having one hole such that $z_2' = (x'x'')^{n_1} x_1' x'' (x'x'')^{n_2}$ with $x_1' \subset x'$ and $n_1 + n_2 + 1 = n$, or there are integers n_1, n_2 and word x_2' having one hole such that $z_2' = (x'x'')^{n_1} x' x_2' (x'x'')^{n_2}$ with $x_2' \subset x''$ and $n_1 + n_2 + 1 = n$. Because $z_2' \subset z_2$, it implies that either $z_2 = (x'x'')^{n_1} x_1 x'' (x'x'')^{n_2}$ with $x_1' \subset x_1$, or $z_2 = (x'x'')^{n_1} x' x_2 (x'x'')^{n_2}$ with $x_2' \subset x_2$.

We treat the second case (a proof for the first case works in a similar manner). Firstly, note that z_5 has $x'x''$ as its prefix because z_1 and z_5 share a prefix of length $|x|$ due to $v \uparrow v'$. Due to the abovementioned presentations of z_2' and z_3, $z_2' z_3 = (x'x'')^{n_1} x' x_2' x' (x''x')^{n_2+p}$. Since $z_1 z_2' z_3 \uparrow z_3 z_4 z_5$, we have either $z_5 = (x''x')^n$ when $n_2 + p \geq n$ or $z_5 \uparrow (x''x')^{n_1-p} x_2' x' (x''x')^{n_2+p}$ otherwise. Therefore, in the first case, or in the second case with $n_1 - p \geq 1$, we have $x'x'' = x''x'$. This means that x' and x'' are powers of a common word z, but this contradicts the primitivity of w. The only case to be considered is the second case with $n_1 - p = 0$. Then $x'x'' \uparrow x_2' x'$. By Lemma 3, there exist full words y', y'' such that $x_2' \subset y'y''$, $x'' = y''y'$, and $x' = (y'y'')^r y'$ for some $r \geq 0$. Recall that $x_2' \subset x''$. Lemma 1 combines these to give $y'y'' = y''y'$. The following argument is the same as the one for the previous case, and leads us to the same contradiction. □

4 The case $|v| \geq 2|w|$

The following theorem treats the case when $|v| \geq 2|w|$.

Theorem 5. *Let ww', vv', and uu' be three squares starting at the same position (not necessarily last occurrences) in a partial word with one hole, such that $|w| < |v| < |u|$, and $(w \wedge w')$ is primitive. If $|v| \geq 2|w|$, then $|w| + |v| \leq |u|$.*

Proof. Let us first assume that the hole is not in ww', and so $ww' = w^2$ and w is a full word (see Figure 2). There exists a nonempty partial word z_1 such that $u = vz_1$. If $|z_1| \geq |w|$, then we are done since $|w| + |v| \leq |u|$. Hence, we consider the case when $|z_1| < |w|$. This implies that $ww \uparrow z_1 w_1 z_2'$, where w_1 is the prefix of length $|w|$ of u' and $|z_1 z_2'| = |w|$. We have that z_2' is nonempty. Since $w_1 \subset w$, it results that $w_1 w_1 \uparrow z_1 w_1 z_2'$ and according to Lemma 4, w_1 is not primitive. Let $w_1 = z_2 z_1'$ for some z_2, z_1' with $|z_2| = |z_2'|$. By letting $w_2 = z_1 z_2$ and $w_2' = z_1' z_2'$, we have that $z_2 w_2'$ is a prefix of u'. Since u' has a prefix compatible with w^2, this gives us that $ww \uparrow z_2 w_2' z_1''$, for some partial word z_1'' satisfying $|z_1''| = |z_1|$. Since $v \uparrow v'$, we get $ww \uparrow w_2 w_2'$ and so $w_2' \subset w$. Hence, $w_2' w_2' \uparrow z_2 w_2' z_1''$, and w_2' is not primitive by Lemma 4.

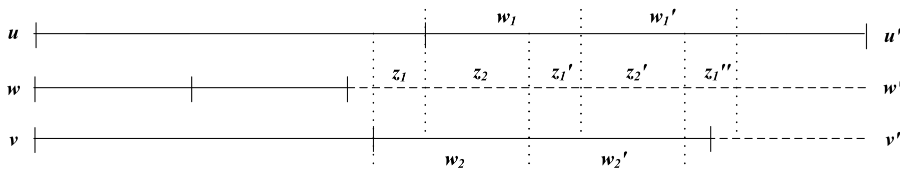

Figure 2: The case when $2|w| \le |v| < |u|$ and the hole is not in ww'

Since $(w \wedge w') = w$ is primitive, $w_1 \subset w$, and w_1 is not primitive, the hole should be in w_1, and a similar argument works to conclude that the hole should be in w_2'. It follows that the hole is in the overlap z_1' of w_1 with w_2'. Since $w_2 = z_1 z_2 = w$, $w_2' = z_1' z_2' \subset w$, and z_2 and z_2' are full, we get $z_2 = z_2'$. We obtain $w = z_1 z_2 = z_2 z_1'' = w_1'$, and by Lemma 3, $z_1 = xy$, $z_1'' = yx$, and $z_2 = (xy)^r x$ for some words x, y and integer $r \ge 0$. Since $w_1 \subset w$, we also obtain $w_1 = z_2 z_1' \subset w = w_1' = z_2 z_1''$, and so $z_1' \subset z_1''$. Since $z_1' \subset z_1 = xy$ and $z_1' \subset z_1'' = yx$, it results from Lemma 1 that x and y are powers of a common word. However, then $(w \wedge w') = w$ would not be primitive, and contradicts with our assumption.

Let us now assume that the hole is in ww'. Suppose towards a contradiction that $|w| + |v| > |u|$. Set the prefix of length $|w|$ of u' as w'' with $w \uparrow w''$. Since $|w| + |v| > |u|$ and $|v| < |u|$, we have $u = vz_1$ for some nonempty z_1 with $|z_1| < |w|$. Since $2|w| \le |v|$ and $v \uparrow v'$, we have $ww' \uparrow z_1 w'' z_2$ for some z_2 where $|z_1 z_2| = |w|$. Note that $(w \wedge w') \subset w \subset w''$ since w'' is full, and that z_2 is nonempty ($z_2 = \varepsilon$ would imply $|z_1| = |w|$ and $|u| = |w| + |v|$). Using Lemma 5, we get a contradiction with the fact that $(w \wedge w')$ is primitive. □

5 The case $|v| < 2|w|$ and the hole is not in ww'

We now start investigating the case when $|v| < 2|w|$.

Theorem 6. *Let ww', vv' and uu' be three squares starting at the same position (not necessarily last occurrences) in a partial word with one hole, such that $|w| < |v| < |u|$ and $(w \wedge w')$ is primitive. If $|v| < 2|w|$ and the hole is not in ww', then $|w| + |v| \le |u|$.*

Proof. Due to the assumption, $w = w'$ holds, and hence, the primitivity of $(w \wedge w')$ means that w is primitive. By Theorem 4, $|u| \ge 2|w|$. For the sake of contradiction, we suppose $|u| < |w| + |v|$, and let us denote $v = wz_1$ and $ww' = vz_2$ for some nonempty full words z_1 and z_2, $u = ww'z_3$ for some partial word z_3, and let z_4 be a nonempty partial word such that z_4 is a prefix of u' and $|uz_4| = |w| + |v|$. Note that $w' = z_1z_2$, $v = z_1z_2z_1$, and $u = z_1z_2z_1z_2z_3$. The length argument easily leads us to $|z_3z_4| = |z_1|$. Let $v' = z_2z_3z_4z_1''$ for some partial word z_1''. Now $v \uparrow v'$ implies that $z_2z_3z_4 \subset z_1z_2$ and $z_1'' \subset z_1$. By Lemma 3, we can see that $z_1z_2z_3z_4$ is strongly $|z_1|$-periodic. Let us denote the prefix of u' of length $|z_1z_2|$ by z_4z_5 for some partial word z_5. Then $u \uparrow u'$ implies that $z_4z_5 \subset z_1z_2$. Combining this with $z_2z_3z_4 \subset z_1z_2$ provides us with $z_2z_3z_4 \uparrow z_4z_5$. Using Lemma 2, we get that $z_2z_3z_4z_5$ is weakly $|z_5|$-periodic. Since $|z_1| = |z_3z_4|$, we can easily observe that $|z_5| = |z_2z_3|$.

Since $v \uparrow v'$, $z_2z_3z_4z_1'' \uparrow z_1z_2z_1$, and from this we obtain $z_2z_3z_4z_1'' \uparrow z_1''z_2z_1$ by weakening, and this implies that $z_2z_3z_4z_1''z_2z_1$ is weakly $|z_1z_2|$-periodic due to Lemma 2.

First, we consider the case when $|z_5| \leq |z_1|$. In this case, z_5 is a prefix of z_1''. We have that $z_2z_3z_4z_5$ is weakly $|z_5|$-periodic, and $z_2z_3z_4z_1''z_2z_1$ is weakly $|z_1z_2|$-periodic. The latter implies that $z_2z_3z_4z_1''$, and hence $z_2z_3z_4z_5$, are weakly $|z_1z_2|$-periodic. By Theorem 3, $z_2z_3z_4z_5$ is strongly $\gcd(|z_5|, |z_1z_2|)$-periodic. Hence, there exists a full word x of length $\gcd(|z_5|, |z_1z_2|)$ such that both $z_2z_3z_4$ and z_5 are contained in powers of x. Since $|z_5| = |z_2z_3|$, both z_2z_3 and z_4 are contained in powers of x. Firstly, if the hole is neither in z_3 nor in z_4, then $w = z_1z_2 = z_2z_3z_4$ is not primitive. Secondly, if the hole is in z_3, then $w = z_1z_2 = z_4z_5$ is not primitive. Thirdly, if the hole is in z_4, then $z_2z_3 = z_5 = x^n$ and $z_4 \subset x^m$ for some positive integers n, m. Set $z_2 = (x'x'')^{n_1}x'$ and $z_3 = x''(x'x'')^{n_2}$, where $x = x'x''$ and where n_1, n_2 are integers satisfying $n_1 + n_2 + 1 = n$. Also set $z_4 = x^{m_1}x_1'x_2'x^{m_2}$, where the hole is in x_1' or x_2' and where m_1 and m_2 are integers satisfying $m_1 + m_2 + 1 = m$. Then $z_1(x'x'')^{n_1}x' = z_1z_2 \uparrow z_4z_5 = x^{m_1}x_1'x_2'x^{m_2}x^n$. Since $|z_1| \geq |z_4|$, this relation enables us to let $z_1 = x^{m_1}x_1x_2x^{m_2}x^{n_2}x_2''$ for some full words x_1, x_2, and x_2'' such that $x_1' \subset x_1$, $x_2' \subset x_2$, and x_2'' is a prefix of x. Since $z_1z_2 \uparrow z_2z_3z_4$, we get $x^{m_1}x_1x_2x^{m_2}x^{n_2}x_2''(x'x'')^{n_1}x' \uparrow x^nx^{m_1}x_1'x_2'x^{m_2}$. By simplification, $x_1 = x'$ and $x_2 = x''$, and $z_1 = x^{n_2+m}x_2''$. Again, using simplification, $x^{m_2}x_2''(x'x'')^{n_1}x' \uparrow x^{n_1}x_1'x_2'x^{m_2}$.

If $n_1 > 0$ and $m_2 > 0$, then $x'x'' = x_2''x' = x''x'$. In this case, x' and x'' are powers of a common word, and so $w = z_1z_2$ is not primitive, a contradiction. The case where $n_1 > 1$ and $m_2 = 0$ similarly follows. If $n_1 = 1$ and $m_2 = 0$, then $x_2''x'x''x' \uparrow x'x''x_1'x_2'$. If the hole is in x_1', then $x_2' = x''$. Here $x'x'' = x_2''x'$ and $x_1'x'' \uparrow x''x'$. The latter implies the existence of words y' and y'' such that $x_1' \subset y'y''$, $x' = y''y'$, and $x'' = (y'y'')^ry'$ for some integer $r \geq 0$. By Lemma 1, since $x_1' \subset y'y''$ and $x_1' \subset x' = y''y'$, we obtain $y'y'' = y''y'$, and so y' and y'' are powers of a common word. Since

$x'x'' = x_2''x'$, x_2'' is also a power of that word, and thus $x_2'' = x''$ and the result follows as above. The proof when the hole is in x_2' is similar. If $n_1 = 0$, then since $z_1z_2 \uparrow z_4z_5$, we get $x^{n_2+m}x_2''x' = z_1z_2 \uparrow z_4z_5 = x^{m_1}x_1'x_2'x^{m_2}x^n$. Looking at the suffixes of z_1z_2 and z_4z_5 of length $|x|$, we get $x_2''x' = x'x''$. We get $w = z_1z_2 = x^{n_2+m}x_2''x' = x^{n_2+m}x'x'' = x^{n_2+m+1} = x^{n_2+m_1+m_2+2}$ is not primitive.

Next, we consider the case when $|z_5| > |z_1|$. In this case, z_1'' is a prefix of z_5. Set $z_5 = z_{1,1}z_{1,2}\cdots z_{1,s}z_6$ for some nonempty partial words $z_{1,1}, z_{1,2}, \ldots, z_{1,s}, z_6$, where $z_{1,1} = z_1''$, $|z_{1,1}| = |z_{1,2}| = \cdots = |z_{1,s}| = |z_1| \geq |z_6|$, and where $s \geq 1$ is an integer. Note that one of $z_{1,1}, \ldots, z_{1,s}, z_6$ may contain the hole. Since $|z_2z_3| = |z_5| = s|z_1| + |z_6| = (s-1)|z_1| + |z_3z_4| + |z_6|$, we get $|z_2| = (s-1)|z_1| + |z_4z_6|$. The compatibility $z_1z_2 \uparrow z_2z_3z_4$ implies that $z_2 = z_1^{s-1}z_6'z_4'$ for some z_4', z_6' such that $z_4 \subset z_4'$ and $|z_6'| = |z_6|$. We obtain $z_1z_1^{s-1}z_6'z_4' = z_1z_2 \uparrow z_2z_3z_4 = z_1^{s-1}z_6'z_4'z_3z_4$, and by simplification $z_1z_6' \uparrow z_6'z_4'z_3$. We deduce that z_6' is a prefix of z_1.

Since $z_2z_3z_4 \uparrow z_4z_5 = z_4z_{1,1}z_{1,2}\cdots z_{1,s}z_6$, let z_4'' be the prefix of z_2 such that $z_4 \subset z_4''$ (note that z_4'' is also a prefix of z_1 since $z_1z_2 \uparrow z_2z_3z_4$). Let $z_{1,1}', z_{1,2}', \ldots, z_{1,s-1}', z_6''$ be such that $z_2 = z_4''z_{1,1}'z_{1,2}'\cdots z_{1,s-1}'z_6''$, where $z_{1,i}' \uparrow z_{1,i}$ for $1 \leq i < s$, and $z_6''z_3z_4 \uparrow z_{1,s}z_6$. Since

$$z_1z_4''z_{1,1}'z_{1,2}'\cdots z_{1,s-1}'z_6'' = z_1z_2 \uparrow z_4z_5 = z_4z_{1,1}z_{1,2}\cdots z_{1,s}z_6$$

we get, using simplification, $z_1z_4'' \uparrow z_4z_{1,1} = z_4z_1''$, $z_{1,i}' \uparrow z_{1,i+1}$ for $1 \leq i < s$, and $z_6'' \uparrow z_6$. The fact that $z_1z_4'' \uparrow z_4z_1''$ implies that z_4'' is compatible with a suffix of z_1''. Since $z_{1,i}'$ is full for every i, $z_{1,i} \uparrow z_{1,i+1}$ for $1 \leq i < s$.

Let us consider the cases when the hole is in z_3 or z_4 or z_5 (the proof when all of z_3, z_4, and z_5 are full is simpler). There are three cases to consider.

Case 1. The hole is not in $z_{1,i}$ for any $1 \leq i \leq s$.

Here, the hole is in z_3 or z_4 or z_6. Moreover, $z_1 = z_{1,1} = z_{1,1}' = z_{1,2} = z_{1,2}' = \cdots = z_{1,s-1} = z_{1,s-1}' = z_{1,s}$, and so $z_5 = z_1^sz_6$ and $z_2 = z_1^{s-1}z_6'z_4' = z_4''z_1^{s-1}z_6''$. We deduce that $z_6'z_4' = z_4''z_6''$.

Firstly, suppose that the hole is not in z_4. Then $z_4 = z_4' = z_4''$ and so $z_6'z_4 = z_4z_6''$. This equation implies that $z_6' = y'y''$, $z_6'' = y''y'$, and $z_4 = (y'y'')^ry'$ for some full words y', y'' and integer $r \geq 0$. Since $z_6''z_3z_4 \uparrow z_{1,s}z_6$, z_6' is a prefix of z_1, and $|z_6''| = |z_6'|$, we obtain $z_6' = y'y'' = y''y' = z_6''$. Recall that $z_2z_3z_4z_5 = z_2z_3z_4z_1^sz_6$ is weakly $|z_5|$-periodic. Recall also that $z_2z_3z_4z_1''z_2z_1$, which is here equal to $z_2z_3z_4z_1z_2z_1 = z_2z_3z_4z_1z_1^{s-1}z_6'z_4z_1$, is weakly $|z_1z_2|$-periodic. Hence, $z_2z_3z_4z_1^sz_6'$, $z_2z_3z_4z_1^sz_6''$, and $z_2z_3z_4z_1^sz_6$ are weakly $|z_1z_2|$-periodic. By Theorem 3, $z_2z_3z_4z_1^sz_6$ is strongly $\gcd(|z_5|, |z_1z_2|)$-periodic. There exists a full word x of length $\gcd(|z_5|, |z_1z_2|)$ such that both $z_2z_3z_4$

and $z_5 = z_1^s z_6$ are contained in powers of x. Since $|z_5| = |z_2z_3|$, both z_2z_3 and z_4 are contained in powers of x. If the hole is in z_3, then $w = z_1z_2 = z_4z_5$ is not primitive. If the hole is in z_6, then $w = z_1z_2 = z_2z_3z_4$ is not primitive.

Secondly, suppose that the hole is in z_4. Then $z_4 \subset z_4'$, $z_4 \subset z_4''$, and $z_6 = z_6''$. Since $z_1z_6' \uparrow z_6'z_4'z_3$, it follows that $z_1z_6'z_4'z_3$ is strongly $|z_1|$-periodic. Also, since $z_1z_6 = z_{1,s}z_6 \uparrow z_6''z_3z_4 = z_6z_3z_4$, we have that $z_1z_6z_3z_4$ is strongly $|z_1|$-periodic. Then both z_6 and z_6' are prefixes of z_1, and since $|z_6| = |z_6'|$, we get that $z_6 = z_6'$. Hence, because $z_6z_4'z_3 = z_1z_6 \uparrow z_6z_3z_4$, it follows that $z_4'z_3 \uparrow z_3z_4$. From Lemmas 3 and 1, there exists a word y such that $z_4 \subset y^{n_1} = z_4'$ and $z_3 = y^{n_2}$, for some integers n_1, n_2.

Moreover, since $|z_1| = |z_4'z_3|$ and $|z_6| \leq |z_1|$, and $z_1z_6z_4'z_3$ is $|z_1|$-periodic, there exist words y' and y'' such that $z_1 = (y'y'')^{n_1+n_2}$, $z_6 = (y'y'')^{n_3}y'$, $z_4'z_3 = (y''y')^{n_1+n_2}$, and $y = y''y'$. We get that

$$z_2 = ((y'y'')^{n_1+n_2})^{s-1}(y'y'')^{n_1+n_3}y' \text{ and } z_5 = ((y'y'')^{n_1+n_2})^s(y'y'')^{n_3}y'.$$

Substituting these into $z_4z_5 \uparrow z_1z_2$ gives

$$z_4((y'y'')^{n_1+n_2})^s(y'y'')^{n_3}y' \uparrow ((y'y'')^{n_1+n_2})^s(y'y'')^{n_1+n_3}y'.$$

Hence, we have $z_4 \subset (y'y'')^{n_1}$. Since we also have $z_4 \subset (y''y')^{n_1}$, Lemma 1 gives that y' and y'' are powers of a common word, and hence, so are z_1 and z_2. However, this contradicts the primitivity of $w = z_1z_2$.

Case 2. The hole is in $z_{1,i}$ for some $1 \leq i \leq s$.

Recall that $z_2 = z_1^{s-1}z_6'z_4' = z_4''z_{1,1}' \cdots z_{1,s-1}'z_6''$. Here, $z_4 = z_4' = z_4''$ and $z_6'' = z_6$. Moreover, $z_1 \supset z_1'' = z_{1,1} = z_{1,1}' = z_{1,2} = z_{1,2}' = \cdots = z_{1,i-1} = z_{1,i-1}' \supset z_{1,i}$ and $z_{1,i} \subset z_{1,i}' = z_{1,i+1} = z_{1,i+1}' = \cdots = z_{1,s-1} = z_{1,s-1}' = z_{1,s}$. Using the facts that $z_2 = z_1^{s-1}z_6'z_4$ and $z_1z_6' = z_6'z_4z_3$, we deduce that $z_2 = z_1^{s-1}z_6'z_4 = z_1^{s-2}z_1z_6'z_4 = z_1^{s-2}z_6'z_4z_3z_4 = z_1^{s-3}z_1z_6'z_4z_3z_4 = \cdots = z_6'z_4(z_3z_4)^{s-1}$. Note that $z_6'z_4$ is both a prefix and a suffix of z_2. Since z_4 is a prefix of z_2 and z_6 is a suffix of z_2, we get $z_6'z_4 = z_4z_6$. The latter implies the existence of words y' and y'' such that $z_6' = y'y''$, $z_6 = y''y'$, and $z_4 = (y'y'')^r y'$ for some integer $r \geq 0$.

Set $z_1 = z_4z_3'$ for some z_3'. Here $z_2 = z_1 \cdots z_1z_6'z_4 = z_4z_{1,1}' \cdots z_{1,s-1}'z_6$ implies that $z_{1,1}' = \cdots = z_{1,s-2}' = z_3'z_4$, and $z_1z_4z_6 = z_1z_6'z_4 = z_4z_{1,s-1}'z_6$. The latter implies that $z_{1,s-1}' = z_3'z_4$. If $1 \leq i < s$, then since $z_{1,i} \subset z_{1,i}' = z_{1,s-1}' = z_3'z_4$ and $z_{1,i} \subset z_1 = z_4z_3'$, we get $z_3'z_4 = z_4z_3'$ by Lemma 1. If $1 < i = s$, then $z_3'z_4 = z_{1,s-1}' = z_1 = z_4z_3'$. So for $1 \leq i \leq s$, $z_5 = z_1^{i-1}z_{1,i}z_{1,s}^{s-i}z_6 = z_1^{i-1}z_{1,i}z_1^{s-i}z_6$ and $z_2 = z_1^{s-1}z_6'z_4 = z_4z_1^{s-1}z_6$.

Recall that $z_6' = y'y''$ is a prefix of z_1 and $z_6 = y''y'$ contains a prefix of $z_{1,s}$; the latter is due to $z_6z_3z_4 \uparrow z_{1,s}z_6$. Since $z_1 \supset z_1'' \subset z_{1,s}$, we can apply Lemma 1 to these to obtain $z_6' = y'y'' = y''y' = z_6$. Recall also that $z_2z_3z_4z_5 = z_2z_3z_4z_1^{i-1}z_{1,i}z_1^{s-i}z_6$ is weakly $|z_5|$-periodic, and $z_2z_3z_4z_1''z_2z_1 = z_2z_3z_4z_1''z_1^{s-1}z_6'z_4z_1$ is weakly $|z_1z_2|$-periodic. Since $z_6' = z_6$, these imply that $z_2z_3z_4z_1^{i-1}z_{1,i}z_1^{s-i}z_6$ is both weakly $|z_5|$-periodic and weakly $|z_1z_2|$-periodic. By Theorem 3, $z_2z_3z_4z_1^{i-1}z_{1,i}z_1^{s-i}z_6$ is strongly $\gcd(|z_5|, |z_1z_2|)$-periodic. There exists a full word x of length $\gcd(|z_5|, |z_1z_2|)$ such that $z_2z_3z_4$ is a power of x and $z_5 = z_1^{i-1}z_{1,i}z_1^{s-i}z_6$ is contained in a power of x. Since $|z_5| < |z_2z_3z_4|$, it must be the case that $w = z_2z_3z_4 = x^p$ for some $p \geq 2$, a contradiction with the assumption that w is primitive. $\square$

Note that Theorem 6 does not necessarily hold when the hole is in the shortest square. Consider for example, the partial word with one hole

$$aba\diamond ababaaabab\underline{a}babab\underline{\underline{a}}baaababaaab\overline{a}bababaaabababababaaababaaab$$

that has three squares, ww', vv', and uu', starting at position zero:

$$\begin{aligned} w &= aba\diamond ababaaabab \\ w' &= ababababaaabab \\ v &= aba\diamond ababaaababababab \\ v' &= abaaababaaababababab \\ u &= aba\diamond ababaaabababababababaaababaaab \\ u' &= ababababaaababababababaaababaaab \end{aligned}$$

of length 14×2, 20×2, and 32×2, respectively (we have underlined with one line the first letter of w', with two lines the first letter of v', and have overlined the first letter of u'). Note that the condition $|w| < |v| < 2|w| \leq |u| < |w| + |v|$ holds. Here, the hole is in ww' and $(w \wedge w') = w$ is primitive (actually, all six words w, w', v, v', u, and u' are primitive).

6 The case $|v| < 2|w|$ and the hole is in ww'

Let ww', vv' and uu' be three squares starting at the same position (not necessarily last occurrences) in a partial word with one hole, such that $|w| < |v| < |u|$ and $(w \wedge w')$ is primitive. In this section, we continue investigating the case where $|v| < 2|w|$, and since the case when the hole is not in ww' has been already investigated in the previous section, in this section we investigate the other case when the hole is in ww'. We will describe precisely the situations when $|w| + |v| \leq |u|$. We refer to Figure 3.

We have that both w and w' are primitive. By Theorem 4, we have $|u| \geq 2|w|$. For the sake of contradiction, we suppose that $|u| < |w| + |v|$. Since $|w| < |v| < 2|w| \leq |u|$, let us denote $v = wz_1$ and $ww' = vz_2$ for some

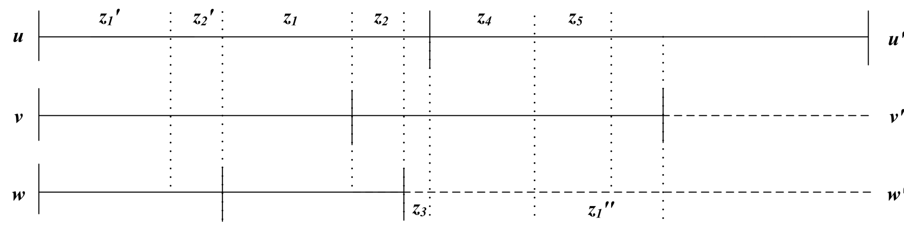

Figure 3: The case when $|w| < |v| < 2|w| \leq |u|$ and the hole is in ww'

nonempty partial words z_1 and z_2, $u = ww'z_3$ for some full word z_3, and let z_4 be a nonempty full word such that z_4 is a prefix of u' and $|uz_4| = |w|+|v|$. We have $w' = z_1z_2$, $w = z'_1z'_2$, $v = z'_1z'_2z_1$, and $u = z'_1z'_2z_1z_2z_3$, where $z'_i \uparrow z_i$ for $i \in \{1, 2\}$. It follows easily that $|z_3z_4| = |z_1|$. Since $v \uparrow v'$, we get that $z'_1z'_2 \uparrow z_2z_3z_4$, and by looking at the prefixes of length $|w|$ of u and u', there exists a partial word z_5, with $|z_5| = |z_2z_3|$, such that z_4z_5 is a prefix of u' and $z'_1z'_2 \uparrow z_4z_5$. Let z''_1 be such that $z_4z''_1$ is a prefix of u' of length $|z_4z_1|$. Since $v \uparrow v'$, $z'_1z'_2z_1 \uparrow z_2z_3z_4z''_1$, and by simplification $z_1 \uparrow z''_1$.

Recall that the hole is in ww'. There are four cases to be considered: Lemma 6 treats the case when the hole is in z_1, Lemma 7 does so when the hole is in z'_2, Lemma 8 when the hole is in z'_1, and Lemma 9 when the hole is in z_2.

Lemma 6. *Let ww', vv', and uu' be three squares starting at the same position (not necessarily last occurrences) in a partial word with one hole, such that $|w| < |v| < |u|$ and $(w \wedge w')$ is primitive. If $|v| < 2|w|$ and the hole is in z_1, then $|w| + |v| \leq |u|$.*

Proof. Here $z_1 \subset z'_1$, $z_1 \subset z''_1$, and $z_2 = z'_2$. Since $v \uparrow v'$, $z'_1z_2 \uparrow z_2z_3z_4$, which implies that $z'_1z_2z_3z_4$ is strongly $|z_1|$-periodic by Lemma 3. Since $u \uparrow u'$, $z'_1z_2 \uparrow z_4z_5$. Since the hole is in z_1, the compatibilities $z'_1z_2 \uparrow z_2z_3z_4$ and $z'_1z_2 \uparrow z_4z_5$ mean that $z'_1z_2 = z_2z_3z_4 = z_4z_5$. Using Lemma 3, we get that $z_2z_3z_4z_5$ is strongly $|z_5|$-periodic. Since $v' \uparrow v$, $z_2z_3z_4z''_1 \uparrow z'_1z_2z_1$. By weakening, $z_2z_3z_4z_1 \uparrow z_1z_2z_1$. By Lemma 2, $z_2z_3z_4z_1z_2z_1$ is weakly $|z_1z_2|$-periodic.

First, we consider the case when $|z_5| \leq |z_1|$. In this case, z_5 is a prefix of z''_1. Let z'_5 be the prefix of z_1 such that $z'_5 \subset z_5$. Since $z_2z_3z_4z_5$ is strongly $|z_5|$-periodic, so is $z_2z_3z_4z'_5$. Also, since $z_2z_3z_4z_1z_2z_1$ is weakly $|z_1z_2|$-periodic, so is $z_2z_3z_4z'_5$. By Theorem 3, $z_2z_3z_4z'_5$ is strongly $\gcd(|z_5|, |z_1z_2|)$-periodic. There exists a full word x of length $\gcd(|z_5|, |z_1z_2|)$ such that $z_2z_3z_4$

is a power of x and z_5' is contained in a power of x. Since $|z_5| < |z_2z_3z_4|$, we obtain that $w = z_1'z_2 = z_2z_3z_4 = x^p$ for some $p \geq 2$, a contradiction with the fact that w is primitive.

Next, we consider the case when $|z_5| > |z_1|$. In this case, z_1'' is a prefix of z_5. Set $z_5 = z_{1,1}z_{1,2}\cdots z_{1,s}z_6$ for some nonempty full words $z_{1,1}, z_{1,2}, \ldots, z_{1,s}, z_6$, where $z_{1,1} = z_1''$, $|z_{1,1}| = |z_{1,2}| = \cdots = |z_{1,s}| = |z_1| \geq |z_6|$, and where $s \geq 1$ is an integer. Since $|z_2z_3| = |z_5| = s|z_1| + |z_6| = (s-1)|z_1| + |z_3z_4| + |z_6|$, we get $|z_2| = (s-1)|z_1| + |z_4z_6|$. The equation $z_1'z_2 = z_2z_3z_4$ implies that $z_2 = (z_1')^{s-1}z_6'z_4$ for some z_6' such that $|z_6'| = |z_6|$. We obtain $z_1'(z_1')^{s-1}z_6'z_4 = z_1'z_2 = z_2z_3z_4 = (z_1')^{s-1}z_6'z_4z_3z_4$, and by simplification $z_1'z_6' = z_6'z_4z_3$. We deduce that z_6' is a prefix of z_1'.

The equation $z_2z_3z_4 = z_4z_5 = z_4z_{1,1}z_{1,2}\cdots z_{1,s}z_6$, with $|z_2| = (s-1)|z_1| + |z_4z_6|$, enables us to let $z_2 = z_4z_{1,1}z_{1,2}\cdots z_{1,s-1}z_6''$. It is easy to observe that $z_6''z_3z_4 \uparrow z_{1,s}z_6$ holds. The equation

$$z_1'z_4z_{1,1}z_{1,2}\cdots z_{1,s-1}z_6'' = z_4z_{1,1}z_{1,2}\cdots z_{1,s}z_6$$

holds, and hence, we can observe that $z_1'z_4 = z_4z_1''$, $z_1'' = z_{1,1} = \cdots = z_{1,s}$, and $z_6'' \uparrow z_6$. From these, we can immediately obtain that $z_5 = (z_1'')^s z_6$ and $z_6''z_3z_4 = z_1''z_6$.

From $z_2 = (z_1')^{s-1}z_6'z_4 = z_4(z_1'')^{s-1}z_6''$, $z_6'z_4 = z_4z_6''$ immediately follows. So there exist y' and y'' such that $z_6' = y'y''$, $z_6'' = y''y'$, and $z_4 = (y'y'')^r y'$ for some integer $r \geq 0$. We claim that $z_1' = z_1''$. Indeed, since $z_1'z_4 \uparrow z_4z_1''$, there exist x' and x'' such that $z_1' = x'x''$, $z_1'' = x''x'$, and $z_4 = (x'x'')^t x'$ for some integer $t \geq 0$. So $z_1 \subset z_1' = x'x''$ and $z_1 \subset z_1'' = x''x'$ and by Lemma 1, $z_1' = x'x'' = x''x' = z_1''$. The prefix of length $|z_6|$ of z_1'' is $z_6'' = y''y'$ (recall that $z_6''z_3z_4 \uparrow z_1''z_6$). Since z_6' is a prefix of z_1', the prefix of length $|z_6|$ of z_1' is $y'y''$. So the prefix of length $|z_6|$ of z_1 is contained in both $y''y'$ and $y'y''$. By Lemma 1, we obtain $z_6' = y'y'' = y''y' = z_6''$. We have that $z_2z_3z_4z_5 = z_2z_3z_4(z_1'')^s z_6$, and hence $z_2z_3z_4z_1(z_1')^{s-1}z_6$, is strongly $|z_5|$-periodic, and $z_2z_3z_4z_1z_2z_1 = z_2z_3z_4z_1(z_1')^{s-1}z_6'z_4z_1$, and hence $z_2z_3z_4z_1(z_1')^{s-1}z_6'$, $z_2z_3z_4z_1(z_1')^{s-1}z_6''$, and $z_2z_3z_4z_1(z_1')^{s-1}z_6$, are weakly $|z_1z_2|$-periodic. By Theorem 3, $z_2z_3z_4z_1(z_1')^{s-1}z_6$ is strongly $\gcd(|z_5|, |z_1z_2|)$-periodic. There exists a full word x of length $\gcd(|z_5|, |z_1z_2|)$ such that both $z_2z_3z_4$ and $z_1(z_1')^{s-1}z_6$ are contained in powers of x. Since $|z_1(z_1')^{s-1}z_6| < |z_2z_3z_4|$, it must be the case that $w = z_1'z_2 = z_2z_3z_4 = x^p$ for some $p \geq 2$, a contradiction with the fact that w is primitive. □

Lemma 7. *Let ww', vv', and uu' be three squares starting at the same position (not necessarily last occurrences) in a partial word with one hole, such* *that* $|w| < |v| < |u|$ *and* $(w \wedge w')$ *is primitive.*

1. *If* $|v| < 2|w|$*, the hole is in* z_2'*, and* $|z_5| \leq |z_1|$*, then* $|w| + |v| \leq |u|$*.*

2. *If* $|v| < 2|w|$ *and* $|z_5| > |z_1|$*, then let* s *be the largest positive integer such that* $|z_5| - s|z_1|$ *is positive. If the hole is in the suffix of length* $|z_5| - s|z_1|$ *of* z_2'*, then* $|w| + |v| \leq |u|$*.*

Proof. Here $z_1 = z_1' = z_1''$ and $z_2' \subset z_2$. Since $v \uparrow v'$, $z_1 z_2' \uparrow z_2 z_3 z_4$. By weakening, we get $z_1 z_2' \uparrow z_2' z_3 z_4$, and so $z_1 z_2' z_3 z_4$ is weakly $|z_1|$-periodic by Lemma 2. Since $u \uparrow u'$, $z_1 z_2' \uparrow z_4 z_5$. The latter and the fact that $|z_1| = |z_3 z_4|$ imply that there exist z_2'', z_3' such that $z_1 = z_4 z_3'$ and $z_5 = z_3' z_2''$, with $|z_3'| = |z_3|$ and $z_2'' \uparrow z_2'$. Since $z_2 z_3 z_4 \uparrow z_1 z_2'$, we get $z_2 z_3 z_4 \uparrow z_4 z_3' z_2'$. Using Lemma 3, we get that $z_2 z_3 z_4 z_3' z_2'$ is strongly $|z_5|$-periodic and so is $z_2' z_3 z_4 z_3' z_2'$. Since $v' \uparrow v$, $z_2 z_3 z_4 z_1 \uparrow z_1 z_2' z_1$. By Lemma 3, $z_2 z_3 z_4 z_1 z_2' z_1$ is strongly $|z_1 z_2|$-periodic.

First, we consider the case when $|z_5| \leq |z_1|$. In this case, z_5 is a prefix of z_1. We have that $z_2 z_3 z_4 z_3' z_2'$ is strongly $|z_5|$-periodic, and $z_2 z_3 z_4 z_1 z_2' z_1$ is strongly $|z_1 z_2|$-periodic. The latter implies that $z_2 z_3 z_4 z_1$ is strongly $|z_1 z_2|$-periodic and so are $z_2 z_3 z_4 z_5$, $z_2 z_3 z_4 z_3' z_2''$, and $z_2 z_3 z_4 z_3' z_2'$. By Theorem 3, $z_2 z_3 z_4 z_3' z_2'$ is strongly $\gcd(|z_5|, |z_1 z_2|)$-periodic. There exists a full word x of length $\gcd(|z_5|, |z_1 z_2|)$ such that $z_2 z_3 z_4$ is a power of x. Since $|x| = \gcd(|z_5|, |z_1 z_2|) \leq |z_5| \leq |z_1| < |z_2 z_3 z_4|$, it must be the case that $w = z_1 z_2' \subset z_2 z_3 z_4 = x^p$ for some $p \geq 2$, a contradiction with the fact that w is primitive.

Next, we consider the case when $|z_5| > |z_1|$. In this case, z_1 is a prefix of z_5. Set $z_5 = z_{1,1} z_{1,2} \cdots z_{1,s} z_6$ for some nonempty words $z_{1,1}, z_{1,2}, \ldots, z_{1,s}, z_6$, where $z_{1,1} = z_1$, $|z_{1,1}| = |z_{1,2}| = \cdots = |z_{1,s}| = |z_1| \geq |z_6|$, and where $s \geq 1$ is an integer. Since $|z_2 z_3| = |z_5| = s|z_1| + |z_6| = (s-1)|z_1| + |z_3 z_4| + |z_6|$, we get $|z_2| = (s-1)|z_1| + |z_4 z_6|$. Since $z_4 z_3' z_2' = z_1 z_2' \uparrow z_4 z_5 = z_4 z_{1,1} z_{1,2} \cdots z_{1,s} z_6$, we get that z_3' is a prefix of $z_{1,1} = z_1$, and let z_4' be such that $z_1 = z_4 z_3' = z_3' z_4'$. Also let z_6' be such that $z_2' = z_4' z_{1,2} z_{1,3} \cdots z_{1,s} z_6'$, where $z_6' \uparrow z_6$. Similarly, since $z_2' \uparrow z_2$, let z_6'' be such that $z_2 = z_4' z_{1,2} z_{1,3} \cdots z_{1,s} z_6''$, where $z_6'' \uparrow z_6'$. Substituting these into $z_1 z_2' \uparrow z_2 z_3 z_4$ gives

$$z_4 z_3' z_4' z_{1,2} \cdots z_{1,s} z_6' \uparrow z_4' z_{1,2} \cdots z_{1,s} z_6'' z_3 z_4.$$

From this relation, we can observe that $z_4 = z_4'$ and $z_1 = z_3' z_4' = z_{1,2} = \cdots = z_{1,s}$. Then, $z_5 = z_1^s z_6$, $z_2 = z_4 z_1^{s-1} z_6''$, and $z_2' = z_4 z_1^{s-1} z_6'$. Here $z_1 = z_4 z_3' = z_3' z_4$, and since $z_1^s z_6 = z_5 = z_3' z_2''$, we get that $z_2'' = z_4 z_1^{s-1} z_6$.

Since z_4 is a suffix of z_1, we can write $z_2' = z_4 z_1^{s-1} z_6' = z_1^{s-1} z_4 z_6'$. Moreover, since $z_1 z_2' \uparrow z_2 z_3 z_4$, we get that z_4 is compatible with a suffix of z_2'. Let

z_6''' of length $|z_6|$ be such that $z_2' = z_1^{s-1} z_4 z_6' = z_1^{s-1} z_6''' \cdots$ and so $z_6''' z_4 \uparrow z_4 z_6'$. There exist y' and y'' such that $z_6''' \subset y'y''$, $z_6' \subset y''y'$, and $z_4 = (y'y'')^r y'$ for some integer $r \geq 0$. Recall that $z_6'' z_3 z_4 \uparrow z_{1,s-1}' z_6' = z_1 z_6'$, and so z_6'' is a prefix of z_1.

Firstly, let us consider $r > 0$. Then $z_1 z_2' \uparrow z_4 z_5$ implies that the prefix of z_1 of length $|z_6|$, which is z_6'', is $y'y''$ so that we have $z_6' \subset z_6'' = y'y''$. With $z_6' \subset y''y'$, this implies $y'y'' = y''y'$, that is, y' and y'' are powers of a common word z. This implies that $z_6''' \uparrow z_6'$, and since z_6' has a hole, we can write more precisely that $z_6' \subset z_6'''$. We have that $z_2 z_3 z_4 z_3' z_2' = z_2 z_3 z_4 z_1^s z_6'$ is strongly $|z_5|$-periodic, and $z_2 z_3 z_4 z_1 z_2' z_1 = z_2 z_3 z_4 z_1 z_1^{s-1} z_6''' \cdots$, and hence $z_2 z_3 z_4 z_1^s z_6'''$ and $z_2 z_3 z_4 z_1^s z_6'$, are strongly $|z_1 z_2|$-periodic. By Theorem 3, $z_2 z_3 z_4 z_1^s z_6'$ is strongly $\gcd(|z_5|, |z_1 z_2|)$-periodic. There exists a full word x of length $\gcd(|z_5|, |z_1 z_2|)$ such that both $z_2 z_3 z_4$ and $z_1^s z_6'$ are contained in powers of x. We deduce that $w = z_1 z_2' \subset z_2 z_3 z_4 = x^m$ for some integer $m \geq 2$, a contradiction with the fact that w is primitive (note that $|z_1 z_2'| = |z_4 z_5| > |z_5| \geq |x|$, and so $m \geq 2$).

Secondly, let us consider $r = 0$. Then $z_4 = y'$, and since $z_6' \subset y''y'$, we have $|z_6'| \geq |z_4|$. Since $z_1 = z_4 z_3' = z_3' z_4$, it follows that there exists a primitive word x such that $z_4 = x^{k_1}$ and $z_3' = x^{k_2}$ for some positive integers k_1, k_2. Because $z_1 z_6' \uparrow z_6'' z_3 z_4$, we get that z_6'' is a prefix of $x^{k_1+k_2}$ and also that z_6' has a suffix compatible with x. The former leads to the existence of words x' and x'' such that $x = x'x''$ and $z_6' \subset z_6'' = x^{k_3} x'$, for some integer k_3. Hence, the suffix of length $|x|$ of z_6' is contained in both $x'x''$ and $x''x'$. Using Lemma 1, we get that x', x'', z_1, z_4, and z_6' are all contained in powers of a common word. This is a contradiction with the fact that $(w \wedge w') = w = z_1 z_2' = z_1 z_4 z_1^{s-1} z_6'$ is primitive. □

Note that Lemma 7 does not necessarily hold when $|z_5| > |z_1|$ and the hole is in the prefix of length $|z_4|$ of z_2'. As a counterexample, consider

$$\begin{array}{rclrcl} |w| &=& 6 & ww' &=& aa\diamond abaaaaaba \\ |v| &=& 8 & vv' &=& aa\diamond abaaaaababaaa \\ |u| &=& 13 & uu' &=& aa\diamond aba\underline{a}a\underline{\underline{a}}abab\bar{a}aaabaaaaabab \end{array}$$

Here $z_1 = aa, z_2 = aaba, z_3 = b, z_4 = a, z_5 = aaaba, z_1' = aa$, and $z_2' = \diamond aba$.

Now, let us consider the case when the hole is in z_1'. Since $u \uparrow u'$, $z_1' z_2' = z_1' z_2 \uparrow z_4 z_5$. The latter and the fact that $|z_5| = |z_2 z_3|$ imply that there exist z_4', z_3', and z_3'' such that $z_1' = z_4' z_3''$ and $z_5 = z_3' z_2$, with $|z_3'| = |z_3|$, $z_4' \uparrow z_4$, and $z_3'' \uparrow z_3'$.

Lemma 8. *Let ww', vv', and uu' be three squares starting at the same position (not necessarily last occurrences) in a partial word with one hole, such* *that $|w| < |v| < |u|$ and $(w \wedge w')$ is primitive.*

1. *If $|v| < 2|w|$ and the hole is in z_4', then $|w| + |v| \leq |u|$.*

2. *If $|v| < 2|w|$, the hole is in z_3'', and $|z_5| \leq |z_1|$, then $|w| + |v| \leq |u|$.*

Proof. Here $z_1' \subset z_1 = z_1''$ and $z_2 = z_2'$. Since $v \uparrow v'$, $z_1'z_2 = z_4'z_3''z_2 \uparrow z_2z_3z_4$, which implies that $z_1'z_2z_3z_4$ is strongly $|z_1|$-periodic by Lemma 3.

First, let us consider Statement 1. The hole being in z_4' implies that $z_3'' = z_3'$. Using weakening we get that $z_2z_3z_4' \uparrow z_4'z_3'z_2$ and using Lemma 2, we get that $z_2z_3z_4'z_3'z_2$ is weakly $|z_5|$-periodic. Recall that $z_1'z_2z_3z_4 = z_4'z_3'z_2z_3z_4$, and thus $z_4'z_3'z_2z_3z_4'$, is strongly $|z_1|$-periodic. Since $v' \uparrow v$, $z_2z_3z_4z_1 \uparrow z_1'z_2z_1$. By weakening, $z_2z_3z_4z_1' \uparrow z_1'z_2z_1$. By Lemma 2, $z_2z_3z_4z_1'z_2z_1$ is weakly $|z_1z_2|$-periodic.

Firstly, assume that $|z_5| = |z_1|$. Then $z_5 = z_1$, and $|z_4| = |z_2|$. The latter and the fact that $z_2z_3z_4z_1'z_2z_1$ is weakly $|z_1z_2|$-periodic imply that $z_4 = z_2$. Since $z_1'z_2 \uparrow z_4z_5 = z_2z_5 = z_2z_1$, there exist y' and y'' such that $z_1' \subset y'y''$, $z_1 = y''y'$, and $z_2 = (y'y'')^r y'$ for some integer $r \geq 0$. By Lemma 1, since $z_1' \subset y'y''$ and $z_1' \subset z_1 = y''y'$, we get $y'y'' = y''y'$ and so there exists a word z such that both y' and y'' are powers of z. Hence, $w' = z_1z_2$ is not primitive, a contradiction.

Secondly, assume that $|z_5| < |z_1|$. Then z_5 is a prefix of z_1 and $|z_2| < |z_4|$. Since $z_4'z_3' = z_1' \subset z_1$, set $z_1 = z_4''z_3'$ for some z_4'' satisfying $z_4' \subset z_4''$. Since $z_5 = z_3'z_2$ is a prefix of z_1, let z_4''' be such that $z_1 = z_4''z_3' = z_3'z_4'''$ (note that z_2 is a prefix of z_4'''). But since $z_4' \uparrow z_4''$ and $z_4'z_3' \uparrow z_4''z_3' = z_3'z_4'''$, we have by Lemma 2, $z_4'z_3'z_4'''$ is strongly $|z_4|$-periodic. Since z_2 is a prefix of z_4''', it follows that $z_4'z_3'z_2$ is strongly $|z_4|$-periodic. Recall that $z_2z_3z_4'z_3'z_2$ is weakly $|z_2z_3|$-periodic, and so $z_4'z_3'z_2$ is as well. Hence, $w = z_1'z_2 = z_4'z_3'z_2$ is strongly $|x| = \gcd(|z_4|, |z_2z_3|)$-periodic for some word x, implying that $w \subset x^m$ for some integer $m \geq 2$, which is a contradiction with the fact that w is primitive.

Thirdly, assume that $|z_5| > |z_1|$. Then z_1 is a prefix of z_5, and since $|z_5| = |z_3z_2| > |z_1| = |z_3z_4|$, there exists z_4'' such that $z_1 = z_3'z_4''$ and z_4'' is a prefix of z_2. Since $z_1' \uparrow z_1$, it follows that $z_4'z_3' \uparrow z_3'z_4''$, and hence, $z_4'z_3'z_4''$ is strongly $|z_4|$-periodic. Since $z_2z_3z_4 \uparrow z_4'z_3'z_2$ and z_4'' is a prefix of z_2, by simplification, we obtain that $z_4'' \uparrow z_4'$, or more precisely, $z_4' \subset z_4''$. Thus $z_4'z_3'z_4''$ is also strongly $|z_3z_4|$-periodic, it follows that $z_4'z_3'z_4''$ is strongly $|x| = \gcd(|z_4|, |z_3z_4|) = \gcd(|z_3|, |z_3z_4|)$-periodic for some word x. Hence, $z_3' = x^m$ and $z_4'' = x^n$ for some integers m, n, and $z_1 = z_3'z_4'' = x^{m+n}$.

Since $z_4'z_3'z_2z_3z_4$ is strongly $|z_1|$-periodic, and z_4'' is a prefix of z_2, it follows that there exist an integer $p \geq n$ and a prefix x' of x, such that $z_2 = x^p x'$ and

$x'z_3z_4$ is a prefix of x^{m+n+1}. Hence, there exists a factorization $x = x'x''$ such that $z_3 = (x''x')^m$ and $z_4 = (x''x')^n$. But $z_4' \subset z_4'' = (x'x'')^n$ and $z_4' \subset z_4 = (x''x')^n$, and so the prefix of length $|x|$ of z_4' is contained in both $x'x''$ and $x''x'$. According to Lemma 1, we get that $x'x'' = x''x'$, hence x' and x'' are both powers of a common word. Furthermore, z_1' and z_2 are contained in powers of that word and the conclusion follows since $w = z_1'z_2$.

Now, let us consider Statement 2. The hole being in z_3'' implies that $z_4' = z_4$. Here $z_2z_3z_4 \uparrow z_4z_3''z_2$, and using Lemma 3, we get that $z_2z_3z_4z_3''z_2$ is strongly $|z_5|$-periodic. Recall that $z_1'z_2z_3z_4 = z_4z_3''z_2z_3z_4$ is strongly $|z_1|$-periodic. Since $v' \uparrow v$, $z_2z_3z_4z_1 \uparrow z_1'z_2z_1$. By weakening, $z_2z_3z_4z_1' \uparrow z_1'z_2z_1$. By Lemma 2, $z_2z_3z_4z_1'z_2z_1$ is weakly $|z_1z_2|$-periodic.

Firstly, assume that $|z_5| = |z_1|$. Then $z_5 = z_1$ and $|z_2| = |z_4|$. The latter and the fact that $z_2z_3z_4z_1'z_2z_1$ is weakly $|z_1z_2|$-periodic imply that $z_4 = z_2$. Since $z_1'z_2 \uparrow z_4z_5 = z_2z_5 = z_2z_1$, there exist y' and y'' such that $z_1' \subset y'y''$, $z_1 = y''y'$, and $z_2 = (y'y'')^r y'$ for some integer $r \geq 0$. By Lemma 1, since $z_1' \subset y'y''$ and $z_1' \subset z_1 = y''y'$, we get $y'y'' = y''y'$ and so there exists a word z such that both y' and y'' are powers of z. Hence, $w' = z_1z_2$ is not primitive, a contradiction.

Secondly, assume that $|z_5| < |z_1|$. Then z_5 is a prefix of z_1 and $|z_2| < |z_4|$, and since $z_4z_3'' = z_1' \subset z_1$, set $z_1 = z_4z_3'''$ for some z_3''' satisfying $z_3'' \subset z_3'''$. Since $z_5 = z_3'z_2$ is a prefix of z_1, let z_4'' be such that $z_1 = z_4z_3''' = z_3'z_4''$ (note that z_2 is a prefix of z_4''). But since $z_3'' \uparrow z_3'$ and $z_4z_3'' \uparrow z_4z_3''' = z_3'z_4''$, we have by simplification, $z_4z_3'' \uparrow z_3''z_4''$. By Lemma 2, $z_4z_3''z_4''$ is weakly $|z_4|$-periodic. Since z_2 is a prefix of z_4'', it follows that $z_4z_3''z_2$ is weakly $|z_4|$-periodic. Recall that $z_2z_3z_4z_3''z_2$ is strongly $|z_2z_3|$-periodic, and so $z_4z_3''z_2$ is as well. Hence, $w = z_1'z_2 = z_4z_3''z_2$ is strongly $|x| = \gcd(|z_4|, |z_2z_3|)$-periodic for some word x, implying that $w \subset x^m$ for some integer $m \geq 2$, which is a contradiction with the fact that w is primitive. $\square$

Note that Statement 2 of Lemma 8 does not necessarily hold when $|z_5| > |z_1|$. As a counterexample, consider

$$\begin{array}{rclcrcl} |w| & = & 5 & & ww' & = & a\diamond abaaaaba \\ |v| & = & 7 & & vv' & = & a\diamond abaaaababaaa \\ |u| & = & 11 & & uu' & = & a\diamond aba\underline{a}a\underline{\underline{a}}bab\overline{a}aabaaaabab \end{array}$$

Here $z_1 = aa, z_2 = aba, z_3 = b, z_4 = a, z_5 = aaba, z_1' = a\diamond$, $z_2' = aba$, $z_3' = a$, $z_4' = a$, and $z_3'' = \diamond$.

Lemma 9. *Let ww', vv', and uu' be three squares starting at the same position (not necessarily last occurrences) in a partial word with one hole, such that $|w| < |v| < |u|$ and $(w \wedge w')$ is primitive.*

1. *If $|v| < 2|w|$ and the hole is in z_2 and $|z_5| \leq |z_1|$, then $|w| + |v| \leq |u|$.*

2. *If $|v| < 2|w|$ and $|z_5| > |z_1|$, then let s be the largest positive integer such that $|z_5| - s|z_1|$ is positive. If the hole is in the prefix of length $|z_4|$ of z_2 or in the suffix of length $|z_5| - s|z_1|$ of z_2, then $|w| + |v| \leq |u|$.*

Proof. Here $z_1 = z_1' = z_1''$ and $z_2 \subset z_2'$. Since $v \uparrow v'$, $z_1 z_2' \uparrow z_2 z_3 z_4$. By weakening, $z_1 z_2 \uparrow z_2 z_3 z_4$, and so $z_1 z_2 z_3 z_4$ is weakly $|z_1|$-periodic by Lemma 2. Since $u \uparrow u'$, $z_1 z_2' \uparrow z_4 z_5$. The compatibilities $z_1 z_2' \uparrow z_2 z_3 z_4$, $z_1 z_2' \uparrow z_4 z_5$ along with the fact that $z_1 z_2'$ is full imply that $z_2 z_3 z_4 \subset z_1 z_2'$ and $z_4 z_5 \subset z_1 z_2'$, and so $z_2 z_3 z_4 \uparrow z_4 z_5$. Using Lemma 3, we get that $z_2 z_3 z_4 z_5$ is strongly $|z_5|$-periodic. Since $v' \uparrow v$, $z_2 z_3 z_4 z_1 \uparrow z_1 z_2' z_1$. By Lemma 3, $z_2 z_3 z_4 z_1 z_2' z_1$ is strongly $|z_1 z_2|$-periodic.

First, we consider the case when $|z_5| \leq |z_1|$. In this case, z_5 is a prefix of z_1. We have that $z_2 z_3 z_4 z_5$ is strongly $|z_5|$-periodic and $z_2 z_3 z_4 z_1 z_2' z_1$ is strongly $|z_1 z_2|$-periodic. The latter implies that $z_2 z_3 z_4 z_1$, and hence $z_2 z_3 z_4 z_5$, are strongly $|z_1 z_2|$-periodic. By Theorem 3, $z_2 z_3 z_4 z_5$ is strongly $\gcd(|z_5|, |z_1 z_2|)$-periodic. There exists a full word x of length $\gcd(|z_5|, |z_1 z_2|)$ such that $z_2 z_3 z_4$ is contained in a power of x and z_5 is a power of x. Since $|z_5| = |z_2 z_3|$, both $z_2 z_3$ and z_4 are contained in powers of x. Hence, $w = z_1 z_2' = z_4 z_5 = x^p$ for some $p \geq 2$, a contradiction with the fact that w is primitive.

Next, we consider the case when $|z_5| > |z_1|$. In this case, $z_1'' = z_1$ is a prefix of z_5. Set $z_5 = z_{1,1} z_{1,2} \cdots z_{1,s} z_6$ for some nonempty partial words $z_{1,1}, z_{1,2}, \ldots, z_{1,s}, z_6$, where $z_{1,1} = z_1$, $|z_{1,1}| = |z_{1,2}| = \cdots = |z_{1,s}| = |z_1| \geq |z_6|$, and where $s \geq 1$ is an integer. Since $|z_2 z_3| = |z_5| = s|z_1| + |z_6| = (s-1)|z_1| + |z_3 z_4| + |z_6|$, we get $|z_2| = (s-1)|z_1| + |z_4 z_6|$. Since $z_2 z_3 z_4 \uparrow z_4 z_5 = z_4 z_{1,1} z_{1,2} \cdots z_{1,s} z_6$, let z_4' and z_6' be such that $z_2 = z_4' z_{1,1} z_{1,2} \cdots z_{1,s-1} z_6'$, where $z_4' \uparrow z_4$ and $z_6' z_3 z_4 \uparrow z_{1,s} z_6$. Similarly, since $z_2' \uparrow z_2$, let z_4'' and z_6'' be such that $z_2' = z_4'' z_{1,1} z_{1,2} \cdots z_{1,s-1} z_6''$, where $z_4'' \uparrow z_4'$ and $z_6'' \uparrow z_6'$. Since

$$z_1 z_4'' z_{1,1} z_{1,2} \cdots z_{1,s-1} z_6'' = z_1 z_2' \uparrow z_4 z_5 = z_4 z_{1,1} z_{1,2} \cdots z_{1,s} z_6$$

we get, using simplification, $z_1 z_4'' \uparrow z_4 z_{1,1} = z_4 z_1$, $z_{1,i} = z_{1,i+1}$ for $1 \leq i < s$, and $z_6'' = z_6$. Since

$$z_1 z_4'' z_{1,1} z_{1,2} \cdots z_{1,s-1} z_6 = z_1 z_2' \uparrow z_2 z_3 z_4 = z_4' z_{1,1} z_{1,2} \cdots z_{1,s-1} z_6' z_3 z_4$$

we get $z_1 z_4'' \uparrow z_4' z_{1,1} = z_4' z_1$. The fact that $z_1 z_4'' \uparrow z_4 z_1$ implies that z_4 is a prefix of z_1 and z_4'' is a suffix of z_1. We have $z_1 = z_{1,1} = z_{1,2} = \cdots = z_{1,s-1} = z_{1,s}$, and so $z_5 = z_1^s z_6$, $z_2 = z_4' z_1^{s-1} z_6'$, and $z_2' = z_4'' z_1^{s-1} z_6$. There are two cases to consider.

Case 1. The hole is in z'_4.

In this case, $z'_6 = z''_6 = z_6$. The compatibility $z_6z_3z_4 = z'_6z_3z_4 \uparrow z_{1,s}z_6 = z_1z_6$ implies that z_6 is a prefix of z_1.

Since $z_1z'_2 \uparrow z_4z_5$, set $z_1 = z_4z'_3$ and $z_5 = z'_3z'_2$ for some z'_3. We deduce $z_1^sz_6 = z_5 = z'_3z'_2 = z'_3z''_4z_1^{s-1}z_6$, and so $z_1 = z_4z'_3 = z'_3z''_4$. The latter implies the existence of words x' and x'' such that $z_4 = x'x''$, $z''_4 = x''x'$, and $z'_3 = (x'x'')^tx'$ for some integer $t \geq 0$. Since the hole is in z'_4, we get $z'_4 \subset z_4 = x'x''$ and $z'_4 \subset z''_4 = x''x'$, and so $z_4 = x'x'' = x''x' = z''_4$, by Lemma 1, and x' and x'' are powers of a common word. Since z_4 is a suffix of z_1, we can write $z'_2 = z_4z_1^{s-1}z_6 = z_1^{s-1}z_4z_6$. Moreover, since $z_1z'_2 \uparrow z_2z_3z_4$, we get thet z_4 is a suffix of z'_2. Let z'''_6 be such that $z'''_6z_4 = z_4z_6$ (note that $z'_2 = z_1^{s-1}z_4z_6 = z_1^{s-1}z'''_6z_4$). So there exist y' and y'' such that $z'''_6 = y'y''$, $z_6 = y''y'$, and $z_4 = (y'y'')^ry'$ for some integer $r \geq 0$. Recall that $z_6z_3z_4 \uparrow z_1z_6$. Then the prefix of length $|z_6|$ of z_1 is $z_6 = y''y'$. Since $z_1z'_2 \uparrow z_4z_5 = z_4z_1^sz_6$, we obtain $y''y'\cdots \uparrow y'y''y'\cdots$ and so $z'''_6 = y'y'' = y''y' = z_6$. We have that $z_2z_3z_4z_5 = z_2z_3z_4z_1^sz_6$ is strongly $|z_5|$-periodic, and $z_2z_3z_4z_1z'_2z_1 = z_2z_3z_4z_1z_1^{s-1}z'''_6z_4z_1$, and hence $z_2z_3z_4z_1^sz'''_6$ and $z_2z_3z_4z_1^sz_6$, are strongly $|z_1z_2|$-periodic. By Theorem 3, $z_2z_3z_4z_1^sz_6$ is strongly $\gcd(|z_5|, |z_1z_2|)$-periodic. There exists a full word x of length $\gcd(|z_5|, |z_1z_2|)$ such that both $z_2z_3z_4$ and $z_5 = z_1^sz_6$ are contained in powers of x. Since $|z_5| = |z_2z_3|$, it must be the case that $z_2z_3 \subset x^n$, $z_4 = x^m$, and $z_5 = x^n$ for some positive integers m, n. We deduce that $w = z_1z'_2 = z_4z_5 = x^{m+n}$, a contradiction with the fact that w is primitive.

Case 2. The hole is in z'_6.

In this case, $z_4 = z'_4 = z''_4$ and $z'_6 \subset z_6 = z''_6$.

Since $z_1z'_2 \uparrow z_4z_5$, set $z_1 = z_4z'_3$ and $z_5 = z'_3z'_2$ for some z'_3. We deduce $z_1^sz_6 = z_5 = z'_3z'_2 = z'_3z_4z_1^{s-1}z_6$, and so $z_1 = z_4z'_3 = z'_3z_4$. Since z_4 is a suffix of z_1, we can write $z'_2 = z_4z_1^{s-1}z_6 = z_1^{s-1}z_4z_6$. Moreover, since $z_1z'_2 \uparrow z_2z_3z_4$, we get that z_4 is a suffix of z'_2. Let z'''_6 be such that $z'''_6z_4 = z_4z_6$ (note that $z'_2 = z_1^{s-1}z_4z_6 = z_1^{s-1}z'''_6z_4$). So there exist y' and y'' such that $z'''_6 = y'y''$, $z_6 = y''y'$, and $z_4 = (y'y'')^ry'$ for some integer $r \geq 0$. Recall that $z'_6z_3z_4 \uparrow z_1z_6$.

Firstly, consider $r > 0$. Then $z'_6 \subset z_6 = y''y'$, and since $z_1z'_2 \uparrow z_4z_5$, the prefix of length $|z_6|$ of z_1 is $y'y''$, and so $z'_6 \subset y'y''$ since the compatibility $z'_6z_3z_4 \uparrow z_{1,s}z_6 = z_1z_6$ implies that z'_6 is compatible with a prefix of z_1. We obtain $z'''_6 = y'y'' = y''y' = z_6$. We have that $z_2z_3z_4z_5 = z_2z_3z_4z_1^sz_6$ is strongly $|z_5|$-periodic, and $z_2z_3z_4z_1z'_2z_1 = z_2z_3z_4z_1z_1^{s-1}z'''_6z_4z_1$, and hence $z_2z_3z_4z_1^sz'''_6$ and $z_2z_3z_4z_1^sz_6$, are strongly $|z_1z_2|$-periodic. By Theorem 3, $z_2z_3z_4z_1^sz_6$ is strongly $\gcd(|z_5|, |z_1z_2|)$-periodic. There exists a full word x

of length $\gcd(|z_5|, |z_1 z_2|)$ such that both $z_2 z_3 z_4$ and $z_5 = z_1^s z_6$ are contained in powers of x. Since $|z_5| = |z_2 z_3|$, it must be the case that $z_2 z_3 \subset x^n$, $z_4 = x^m$, and $z_5 = x^n$ for some positive integers m, n. We deduce that $w = z_1 z_2' = z_4 z_5 = x^{m+n}$, a contradiction with the fact that w is primitive.

Secondly, consider $r = 0$. Then $z_4 = y'$, and since $z_6 = y'' y'$, we have $|z_6| \geq |z_4|$. Moreover, since $z_1 = z_4 z_3' = z_3' z_4$, there exists a primitive word x such that $z_4 = x^{k_1}$, $z_3' = x^{k_2}$, and $z_1 = x^{k_1+k_2}$ for some positive integers k_1, k_2. Since $z_1 z_2 z_3 z_4 = z_1 z_4 z_1^{s-1} z_6' z_3 z_4$ is weakly $|z_1|$-periodic, we have that $z_6' \subset x^{k_3} x'$ for some integer k_3 with $k_1 \leq k_3 \leq k_1 + k_2$ and some prefix x' of x. But, $z_1 z_6 \uparrow z_6' z_3 z_4$ gives us that $z_4 = x^{k_1}$ is compatible with a suffix of z_6, and since $z_6' \subset z_6$, z_4 is compatible with a suffix of z_6'. Hence, if $x = x' x''$, for some word x'', we have that the suffix of length $|x|$ of z_6' is contained in both $x' x''$ and $x'' x'$. Using Lemma 1, we get that x', x'', z_1, z_4, and z_6' are all contained in powers of a common word. This is a contradiction with the fact that $(w \wedge w') = w' = z_1 z_2 = z_1 z_4 z_1^{s-1} z_6'$ is primitive. $\square$

Referring to the notation used in the proof of Lemma 9, note that Statement 2 does not necessarily hold when the hole is in $z_{1,i}'$ for some $1 \leq i < s$. As a counterexample, consider

$$\begin{array}{rclrcl} |w| & = & 7 & ww' & = & aaababaaaa\diamond aba \\ |v| & = & 9 & vv' & = & aaababaaaa\diamond ababaaa \\ |u| & = & 15 & uu' & = & aaababa\underline{a}a\underline{\underline{a}}\diamond abab\overline{a}aababaaaaaabab \end{array}$$

Here $z_1 = aa, z_2 = a\diamond aba, z_3 = b, z_4 = a, z_5 = aababa, z_6 = ba, z_1' = aa$, $z_2' = ababa$, $z_4' = a$, $z_6' = ba$, $z_{1,1} = aa$, $z_{1,2} = ba$, and $z_{1,1}' = \diamond a$ ($s = 2$ here).

We have shown the following theorem.

Theorem 7. *Let ww', vv', and uu' be three squares starting at the same position (not necessarily last occurrences) in a partial word with one hole, such that $|w| < |v| < |u|$ and $(w \wedge w')$ is primitive. If $|v| < 2|w|$ and the hole is in ww' according to one of Lemmas 6, 7, 8, and 9, then $|w| + |v| \leq |u|$.*

7 Conclusion

In this paper, we have proved the following extension of the three-squares theorem to partial words with one hole.

Theorem 8. *Let ww', vv', and uu' be three squares starting at the same position (not necessarily last occurrences) in a partial word with one hole, such that $|w| < |v| < |u|$ and $(w \wedge w')$ is primitive. Then $|u| \geq 2|w|$. Moreover, the following hold:*

1. *If* $|v| \geq 2|w|$*, then* $|w| + |v| \leq |u|$.

2. *If* $|v| < 2|w|$ *and the hole is not in* ww'*, then* $|w| + |v| \leq |u|$.

3. *If* $|v| < 2|w|$ *and the hole is in* ww' *according to one of Lemmas 6, 7, 8, and 9, then* $|w| + |v| \leq |u|$.

Proof. By Theorem 4, $|u| \geq 2|w|$. Statements 1, 2, and 3 follow by Theorems 5, 6, and 7, respectively. □

Corollary 1. *If* ww', vv'*, and* uu' *are three squares starting at the same position (not necessarily last occurrences) in a full word, such that* $|w| < |v| < |u|$ *and* w *is primitive, then* $|w| + |v| \leq |u|$.

Note that the three-squares theorem does not hold for partial words with two holes since, for example, the partial word $a\diamond baabbaabaab\diamond abbaab$ has three squares starting at position zero:

$$(abba)^2, \ (aabaabb)^2, \text{ and } (aabaabbaab)^2$$

of length 4×2, 7×2, and 10×2 respectively. The inequality $|w| + |v| \leq |u|$ does not hold here, but the inequality $|u| \geq 2|w|$ does hold.

Acknowledgement

We would like to thank the referees of a preliminary version of this paper for their very valuable comments and suggestions. Part of this paper was presented at WORDS'09 [10]. We thank Ilkyoo Choi for writing a program that discovered the three examples after Lemmas 7, 8, and 9. We also thank Kristen Wetzler for valuable comments and suggestions. A World Wide Web site has been established at

`http://www.uncg.edu/cmp/research/squares`

for this research.

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