The Three-Squares Lemma for Partial Words with One Hole*

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Abstract. Partial words, or sequences over a finite alphabet that may have do not know symbols or holes, have been recently the subject of much investigation. Several interesting combinatorial properties have been investigated such as the periodic behavior and the counting of distinct squares in partial words. In this paper, we extend the three-squares lemma on words to partial words with one hole. This result provides special information about the squares in a partial word with at most one hole, and puts restrictions on the positions at which periodic factors may occur, which is in contrast with the well known periodicity lemma of Fine and Wilf.

1 Introduction

A *square* in a full word w has the form uu for some factor u of w. A well known problem is the determination of $\sigma(n)$, the maximum number of distinct squares in any full word of length n, where experiment strongly suggests that $\sigma(n) < n$. With this problem progress has been made: Fraenkel and Simpson showed that $\sigma(n) \leq 2n - 2$ [14], a result recently proved somewhat more simply by Ilie [16], then later improved to $\sigma(n) \leq 2n - \Theta(\log n)$ [17]. In fact, it was shown that at each position there are at most two distinct squares whose last occurrence start.

In order to show that $\sigma(n) < n$, one needs to somehow limit to less than one the average number of squares that begin at the positions of w. This requirement draws attention to positions i where two or more squares begin. Is it true that at positions "neighbouring" to i, no squares can begin? Perhaps the most famous theoretical result restricting periodicity is Fine and Wilf's "periodicity lemma" stated as Theorem 1.

Theorem 1 ([13]). *If a full word w has two periods p, q and $|w| \geq p + q - \gcd(p, q)$, then w has also period $\gcd(p, q)$.*

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Unfortunately this theorem provides no special information about the squares, and it puts no restrictions on the positions at which periodic factors may occur. A result that provides such information is the following "three-squares lemma" stated as Theorem 2.

Theorem 2 ([11, 18]). *If w^2, v^2, u^2 are three squares starting at the same position (not necessarily last occurrences) in a full word, such that $v \notin w^*$, $|w| < |v| < |u|$ and w is primitive, then $|w| + |v| \leq |u|$.*

The main result in [12] is essentially a generalization of this result that allows v to be offset by k positions from the start of u^2, and that does not always require complete squares u^2 and v^2, only sufficiently long factors of periods $|u|$ and $|v|$. Moreover, as a corollary, it specifies exactly the periodic behavior in the word. It is hoped that, with the help of such results, it will be possible to establish, or at least make progress with, the conjecture that $\sigma(n) < n$.

The counting of distinct squares in partial words was recently initiated and revealed surprising results [8, 9]. In this case, a square in a partial word over a given alphabet has the form uu' where u' is *compatible* with u, and consequently, such square is compatible with a number of full words over the alphabet that are squares. In [9], it was shown that for partial words with one hole, there may be more than two squares that have their last occurrence starting at the same position. There it was proved that if such is the case, then the hole is in the shortest square. Furthermore, it turns out that the length of the shortest square is at most half the length of the third shortest square [8]. As a result, it was shown that the number of distinct full squares compatible with factors of a partial word with one hole of length n is bounded by $\frac{7n}{2}$.

Although Fine and Wilf's Theorem 1 has extensively been studied in the context of partial words [1, 2, 4, 7, 10, 15, 19, 20], such is not the case of the Three-Squares Theorem 2. In this paper, we prove the three-squares theorem in the context of partial words with one hole.

2 Preliminaries

Fixing a nonempty finite set of letters or an *alphabet* A, a *partial word* u of length $|u| = n$ over A is a partial function $u : \{0, \ldots, n-1\} \to A$. For $0 \leq i < n$, if $u(i)$ is defined, then i belongs to the *domain* of u, denoted by $i \in D(u)$, otherwise i belongs to the *set of holes* of u, denoted by $i \in H(u)$ (a partial word u such that $H(u) = \emptyset$ is also called a *full word*). The unique word of length 0, denoted by ε, is called the *empty* word. For convenience, we will refer to a partial word over A as a word over the enlarged alphabet $A_\diamond = A \cup \{\diamond\}$, where $\diamond \notin A$ represents a hole. For partial words u, v, w, if $w = uv$, then u is a *prefix* of w, denoted by $u \leq w$, and if $v \neq \varepsilon$, then u is a *proper prefix* of w, denoted by $u < w$. If $w = xuy$, then u is a *factor* of w. The set of all words (respectively, nonempty words, partial words, nonempty partial words) over A of finite length is denoted by A^* (respectively, A^+, $A_\diamond^*$, $A_\diamond^+$).

A *strong period* of a partial word u is a positive integer p such that $u(i) = u(j)$ whenever $i, j \in D(u)$ and $i \equiv j \bmod p$. In this case, we call u *strongly p-periodic.* A *weak period* of u is a positive integer p such that $u(i) = u(i+p)$ whenever $i, i+p \in D(u)$. In this case, we call u *weakly p-periodic.* Note that every weakly p-periodic full word is strongly p-periodic but this is not necessarily true for partial words.

Fundamental results on periodicity of full words include the theorem of Fine and Wilf which considers the simultaneous occurrence of different periods in a word. The following theorem extends this result to partial words with one hole.

Theorem 3 ([1]). *Let $w \in A_\diamond^*$ be weakly p-periodic and weakly q-periodic. If $H(w)$ is a singleton and $|w| \geq p + q$, then w is strongly $\gcd(p, q)$-periodic.*

The partial word u is *contained in* the partial word v, denoted by $u \subset v$, provided that $|u| = |v|$, all elements in $D(u)$ are in $D(v)$, and for all $i \in D(u)$ we have that $u(i) = v(i)$. The *greatest lower bound* of a pair of partial words u and v of equal length is the partial word $u \wedge v$ such that $(u \wedge v) \subset u$ and $(u \wedge v) \subset v$, and for all partial words w which satisfy $w \subset u$ and $w \subset v$ we have that $w \subset (u \wedge v)$.

A partial word u is *primitive* if there exists no word v such that $u \subset v^n$ with $n \geq 2$. If u is a nonempty partial word, then there exist a primitive word v and a positive integer n such that $u \subset v^n$. Uniqueness holds for full words but not for partial words as seen with $u = \diamond a$ where $u \subset a^2$ and $u \subset ba$ for distinct letters a, b. Note that if $u \wedge v$ is primitive for some partial words u, v of equal length, then both u and v are primitive.

The partial words u and v are *compatible*, denoted by $u \uparrow v$, provided that there exists w such that $u \subset w$ and $v \subset w$. An equivalent formulation of compatibility is that $|u| = |v|$ and for all $i \in D(u) \cap D(v)$ we have that $u(i) = v(i)$. The following rules are useful for computing with partial words: (1) *Multiplication:* If $u \uparrow v$ and $x \uparrow y$, then $ux \uparrow vy$; (2) *Simplification:* If $ux \uparrow vy$ and $|u| = |v|$, then $u \uparrow v$ and $x \uparrow y$; and (3) *Weakening:* If $u \uparrow v$ and $w \subset u$, then $w \uparrow v$.

The following lemmas will be useful for our purposes.

Lemma 1 ([1]). *Let $x, y \in A^+$ and let $z \in A_\diamond^*$ be such that $H(z)$ is a singleton. If $z \subset xy$ and $z \subset yx$, then $xy = yx$ or x, y are powers of a common word.*

Lemma 2 ([5]). *Let $x, y, z \in A_\diamond^*$ be such that $|x| = |y| > 0$. Then $xz \uparrow zy$ if and only if xzy is weakly $|x|$-periodic.*

Lemma 3 ([6]). *Let $x, y \in A_\diamond^+$ and $z \in A^*$. If $xz \uparrow zy$, then there exist $v, w \in A^*$ and an integer $n \geq 0$ such that $x \subset vw$, $y \subset wv$, and $z = (vw)^n v$. Consequently, if $xz \uparrow zy$, then xzy is strongly $|x|$-periodic.*

If $u = u_1 u_2$ for some nonempty compatible partial words u_1 and u_2, then u is called a *square*. Whenever we refer to a square $u_1 u_2$ it will imply that $u_1 \uparrow u_2$.

3 Three-squares theorem

Unfortunately Fine and Wilf's theorem provides no special information about the squares in a full word, and it puts no restrictions on the positions at which periodic factors may occur. A result that provides such information is the three-squares theorem that was mentioned earlier. The main result in this section is essentially a generalization of this result to partial words with one hole.

We start with a lemma that extends synchronization to such partial words (synchronization is the property that a full word w is primitive if and only if in ww there exist exactly two factors equal to w, namely the prefix and the suffix).

Lemma 4 ([3]). *Let w be a partial word with at most one hole. Then w is primitive if and only if $ww \uparrow xwy$ for some x, y implies $x = \varepsilon$ or $y = \varepsilon$.*

Using the previous lemma, we can easily prove the following.

Lemma 5. *If ww' is a square with one hole such that $ww' \uparrow xw''y$ for some nonempty partial words x, y, and some partial word w'' satisfying $(w \wedge w') \subset w''$, then $(w \wedge w')$ is not primitive.*

Proof. Suppose that $(w \wedge w')$ is primitive. By weakening, we get that $(w \wedge w')(w \wedge w') \uparrow x(w \wedge w')y$. By lemma 4, $x = \varepsilon$ or $y = \varepsilon$, a contradiction. □

Theorem 4. *If ww', vv', uu' are three squares starting at the same position (not necessarily last occurrences) in a partial word with one hole, such that $|w| < 2|w| \leq |v| < |u|$, and $(w \wedge w')$ is primitive, then $|w| + |v| \leq |u|$.*

Proof. Let us first assume that the hole is not in ww', and so $ww' = w^2$ and w is a full word (see Figure 1). There exists a nonempty partial word z_1 such that $u = vz_1$. If $|z_1| \geq |w|$, then we are done since $|w| + |v| \leq |u|$. Hence, we consider the case where $|z_1| < |w|$. But this implies that $ww \uparrow z_1 w_1 z_2'$, where w_1 is the prefix of length $|w|$ of u' and $|z_1 z_2'| = |w|$. We have z_2' nonempty. Since $w_1 \subset w$, it results that $w_1 w_1 \uparrow z_1 w_1 z_2'$ and according to Lemma 4, w_1 is not primitive. But we also have that $z_2 w_2'$ is a prefix of u', for some z_2, w_2' with $|z_2| = |z_2'|$ and $|w_2'| = |w|$. Set $w_2 = z_1 z_2$. Since u' has a prefix compatible with w^2, this gives us that $ww \uparrow z_2 w_2' z_1''$, for some partial word z_1'' satisfying $|z_1''| = |z_1|$. Since $v \uparrow v'$, we get $ww \uparrow w_2 w_2'$ and so $w_2' \subset w$. Hence, $w_2' w_2' \uparrow z_2 w_2' z_1''$, and w_2' is not primitive by Lemma 4. Since $(w \wedge w') = w$ is primitive and $w_1 \subset w$ and w_1 is not primitive, the hole is in w_1, and similarly for w_2'. It follows that the hole is in the overlap z_1' of w_1 with w_2'. Since $w_2 = z_1 z_2 = w$ and $w_2' = z_1' z_2' \subset w$ and z_2, z_2' are full, we get $z_2 = z_2'$. We obtain $w = z_1 z_2 = z_2 z_1'' = w_1'$, and by Lemma 3, $z_1 = xy$, $z_1'' = yx$, and $z_2 = (xy)^r x$ for some words x, y and integer $r \geq 0$. Since $w_1 \subset w$, we also obtain $w_1 = z_2 z_1' \subset w = w_1' = z_2 z_1''$, and so $z_1' \subset z_1''$. Since $z_1' \subset z_1 = xy$ and $z_1' \subset z_1'' = yx$, it results from Lemma 1 that x and y are powers of a common word. We conclude that $(w \wedge w') = w$ is not primitive.

Let us now assume that the hole is in ww'. Suppose towards a contradiction that $|w| + |v| > |u|$. Set the prefix of length $|w|$ of u' as w'' with $w \uparrow w''$. Since

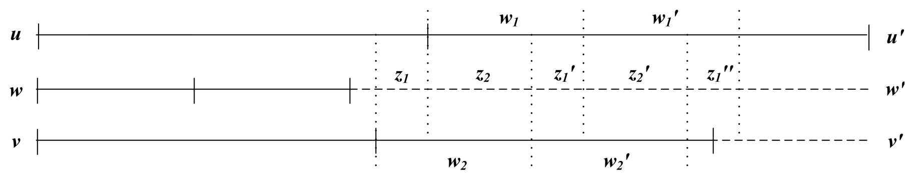

Fig. 1. The case when $|w| < 2|w| \leq |v| < |u|$ and the hole is not in ww'

$|w| + |v| > |u|$ and $|v| < |u|$, we have $u = vz_1$ for some nonempty z_1 with $|z_1| < |w|$. Since $2|w| \leq |v|$ and $v \uparrow v'$, we have $ww' \uparrow z_1 w'' z_2$ for some z_2 where $|z_1 z_2| = |w|$. Note that $(w \wedge w') \subset w \subset w''$ since w'' is full, and that z_2 is nonempty ($z_2 = \varepsilon$ would imply $|z_1| = |w|$ and $|u| = |w| + |v|$). Using Lemma 5, we get a contradiction with the fact that $(w \wedge w')$ is primitive. □

Theorem 5. *If ww', vv', uu' are three squares starting at the same position (not necessarily last occurrences) in a partial word with one hole, such that $|w| < |v| < |u| < 2|w|$, then $(w \wedge w')$ is not primitive.*

Proof. Let us assume that $(w \wedge w')$ is primitive, or that both w and w' are primitive. Since $|w| < |v| < |u|$, let us denote $v = wz_1$ and $u = vz_2$, for some partial words z_1, z_2. Denote $ww' = uz_3$, for some partial word z_3. We have $w' = z_1 z_2 z_3$, $w = z_1' z_2' z_3'$, $v = z_1' z_2' z_3' z_1$ and $u = z_1' z_2' z_3' z_1 z_2$, where $z_i' \uparrow z_i$ for all $i \in \{1, 2, 3\}$. Since $v \uparrow v'$, we get that there exists a partial word z_4 such that $z_2 z_3 z_4$ is a prefix of v' and $|z_4| = |z_1|$, and by looking at the prefixes of length $|w|$ of u and u', we get that there exists a partial word z_5, with $|z_5| = |z_2|$, such that $z_1' z_2' z_3' \uparrow z_3 z_4 z_5$ (see Figure 2).

First, assume that the hole is not in ww'. Here $w = w'$, and consequently $z_1 = z_1', z_2 = z_2', z_3 = z_3'$. Since $v \uparrow v'$, $z_1 z_2 z_3 \uparrow z_2 z_3 z_4$, and since $u \uparrow u'$, $z_1 z_2 z_3 \uparrow$

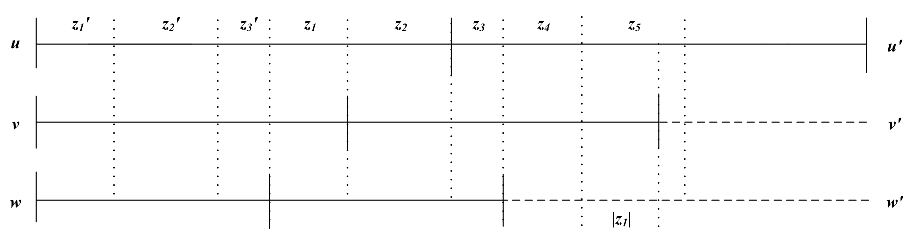

Fig. 2. The case when $|w| < |v| < |u| < 2|w|$

$z_3 z_4 z_5$. The former implies $z_1 z_2 z_3 z_4$ is weakly $|z_1|$-periodic by Lemma 2, while the latter implies that $z_1 z_2 z_3 z_4 z_5$, and hence $z_1 z_2 z_3 z_4$ and $z_2 z_3 z_4 z_5$, are weakly $|z_1 z_2|$-periodic. By Theorem 3, $z_1 z_2 z_3 z_4$ is strongly $\gcd(|z_1|, |z_1 z_2|)$-periodic. Let

x be the prefix of z_1 of length $\gcd(|z_1|, |z_1z_2|)$, and so $z_1 = x^m$ and $z_1z_2 = x^{m+n}$ for some integers $m, n > 0$. Furthermore, since $z_2z_3z_4 \subset z_1z_2z_3$ and $z_3z_4z_5 \subset z_1z_2z_3$, we get that $z_2z_3z_4 \uparrow z_3z_4z_5$, and so $z_2z_3z_4z_5$ is weakly $|z_2|$-periodic by Lemma 2. Hence, $z_2z_3z_4z_5$ is strongly $|x| = \gcd(|z_2|, |z_1z_2|)$-periodic. It must be the case that $z_3 = (x'x'')^p x'$, $z_4 \subset (x''x')^m$ and $z_5 \subset (x''x')^n$, for a factorization $x'x''$ of x and nonnegative integers p, m, n. Let z_5' be the prefix of length $|x|$ of z_5. Since $v \uparrow v'$, we get by simplification, $z_1z_2z_3x \uparrow z_2z_3z_4z_5'$. Using simplification again, we get that $x \uparrow z_5'$, and so $z_5' \subset x'x''$. The latter and the fact that $z_5' \subset x''x'$ give us that x' and x'' are powers of the same word by Lemma 1, which leads to a contradiction with our initial assumption that $w = z_1z_2z_3$ is primitive.

Now, assume that the hole is in ww'. There are six cases to consider: Case 1 (the hole is in z_3), Case 2 (the hole is in z_2), Case 3 (the hole is in z_1), Case 4 (the hole is in z_3'), Case 5 (the hole is in z_2'), and Case 6 (the hole is in z_1'). We treat here Cases 2 and 5.

Case 2. The hole is in z_2.

Hence, $w' = z_1z_2z_3$, $w = z_1z_2'z_3$, where $z_2' \uparrow z_2$, $v = z_1z_2'z_3z_1$ and $u = z_1z_2'z_3z_1z_2$. Since $z_1z_2'z_3 \uparrow z_2z_3z_4$, we get $z_1z_2z_3 \uparrow z_2z_3z_4$ by applying weakening. Using Lemma 2, we get

$$z_1z_2z_3z_4 \text{ is weakly } |z_1|\text{-periodic} \tag{1}$$

Now, since $w' \subset w$, by looking at the prefixes of length $|w|$, $z_2z_3z_4 \subset w$ of v' and $z_3z_4z_5 \subset w$ of u', we get $z_2z_3z_4 \uparrow z_3z_4z_5$. Using Lemma 3, we get

$$z_2z_3z_4z_5 \text{ is strongly } |z_2|\text{-periodic} \tag{2}$$

Finally, for the prefixes of length $|w|$ of u and u', we have $z_1z_2'z_3 = z_3z_4z_5$. Using Lemma 3, it results that

$$z_1z_2'z_3z_4z_5 \text{ is strongly } |z_1z_2|\text{-periodic} \tag{3}$$

From (1) and (3) we get that $z_1z_2z_3z_4$ is weakly $|z_1|$- and $|z_1z_2|$-periodic. Applying Theorem 3, we have $z_1z_2z_3z_4$ strongly $\gcd(|z_1|, |z_1z_2|)$-periodic. Hence, there exists a full word x of length $\gcd(|z_1|, |z_1z_2|)$, such that $z_1 = x^m$ and $z_1z_2 \subset x^{m+n}$, for some integers $m, n > 0$. From (2) and (3) we get that $z_2z_3z_4z_5$ is strongly $|z_2|$- and $|z_1z_2|$-periodic. Applying Theorem 3, $z_2z_3z_4z_5$ is strongly $\gcd(|z_2|, |z_1z_2|)$-periodic. It follows that $z_1z_2z_3z_4z_5$ is strongly $|x|$-periodic.

Because z_1 and z_5 share a prefix of length $\min(|x^m|, |x^n|)$, and $|z_5| = |x^n|$, we get that $z_5 = x^n$. Since $z_3z_4z_5$ is strongly $|x|$-periodic, $|z_5| \geq |x|$ and $|z_4| = |x^m|$, we get that $z_4 = x^m = z_1$. Since $z_1z_2z_3$ is strongly $|x|$-periodic, it results that $z_3 = (x'x'')^p x'$ with $x = x'x''$ and some integer $p \geq 0$. By looking at the prefixes of length $|w|$ of u and u', we notice that $z_1z_2'z_3 = z_3z_1z_5$. This implies that $x'x'' = x''x'$. Results that there exist integers q, r with $q, r \geq 0$ and a word y such that $x' = y^q$ and $x'' = y^r$. But since $w' = z_1z_2z_3$, we get, again, a contradiction with the fact that w' is primitive.

Case 5. The hole is in z_2'.

Looking at the prefixes of length $|w|$ of v and v', we have $z_1z_2'z_3 \uparrow z_2z_3z_4$. Applying weakening and Lemma 2, we get that $z_1z_2'z_3z_4$ is weakly $|z_1|$-periodic. Also, by looking at the prefixes of length $|w|$ of u and u' we get that $z_1z_2'z_3 \uparrow z_3z_4z_5$. By applying Lemma 3, we get that $z_1z_2'z_3z_4z_5$ is strongly $|z_1z_2|$-periodic. Using Theorem 3, it follows that $z_1z_2'z_3z_4$ is strongly $\gcd(|z_1|, |z_1z_2|)$-periodic. Hence, there exists x, such that $z_1 = x^m$ and $z_1z_2' \subset x^{m+n}$, for some positive integers m, n and $|x| = \gcd(|z_1|, |z_1z_2|)$. Hence we have $z_3 = (x'x'')^px'$ and $z_4 = (x''x')^m$, where $x = x'x''$ and $p \geq 0$.

Since the hole is in z_2', either $z_2' = (x'x'')^{n_1}x_1'x''(x'x'')^{n_2}$ where x_1' has a hole, or $z_2' = (x'x'')^{n_1}x'x_2'(x'x'')^{n_2}$ where x_2' has a hole (in either case $n_1+n_2+1 = n$). Because $z_2' \subset z_2$, it implies that $z_2 = (x'x'')^{n_1}x_1x''(x'x'')^{n_2}$ where $x_1' \subset x_1$, or $z_2 = (x'x'')^{n_1}x'x_2(x'x'')^{n_2}$ where $x_2' \subset x_2$. But also, $z_1z_2'z_3 \uparrow z_2z_3z_4$. This is equivalent to one of the following cases: $x^m(x'x'')^{n_1}x_1'x''(x'x'')^{n_2}(x'x'')^px' \uparrow (x'x'')^{n_1}x_1x''(x'x'')^{n_2}(x'x'')^px'(x''x')^m$ when we get $x_1 = x'$, or

$$x^m(x'x'')^{n_1}x'x_2'(x'x'')^{n_2}(x'x'')^px' \uparrow (x'x'')^{n_1}x'x_2(x'x'')^{n_2}(x'x'')^px'(x''x')^m$$

when we get $x_2 = x''$. In either case, $z_2 = (x'x'')^n$.

We treat the second case (the other is similar). Since $z_1z_2'z_3 \uparrow z_3z_4z_5$, there is the possibility that $z_5 = (x''x')^n$ if $n \leq n_2+p$. We leave this case to the reader and assume that $n > n_2 + p$. We get that $z_5 = (x''x')^{n_1-p}x_2x'(x''x')^{n_2+p}$ with $x_2' \subset x_2$. Since $v \uparrow v'$, it follows that z_5 and z_1 share a prefix of length $|x|$, and so z_5 has $x'x''$ as a prefix. There are two cases to consider: (5.1) $x'x'' = x''x'$; and (5.2) $x'x'' = x_2x'$. For (5.1), there exists a full word y such that x' and x'' are powers of y. It follows that z_1, z_2', z_3 are contained in powers of y, implying that $w = z_1z_2'z_3$ is not primitive, a contradiction. For (5.2), $n_1 = p$ and we can denote z_5 as $x'x''(x''x')^{n-1}$. Furthermore, since $z_1z_2'z_3 \uparrow z_3z_4z_5$ we get that $x'(x''x')^{n_1-p}x_2'x'(x''x')^{n_2+p} \uparrow x'x'x''(x''x')^{n-1}$.

Since $n > n_2 + p$, we have $n_1 \geq p$. If $n_1 > p$, then $x'x'' = x''x'$. If $n_1 = p$, then $x_2'x' \uparrow x'x''$. By Lemma 3, there exist full words y', y'' such that $x_2' \subset y'y''$, $x'' = y''y'$, and $x' = (y'y'')^ry'$ for some integer $r \geq 0$. By Lemma 1, since $x_2' \subset y'y''$ and $x_2' \subset x'' = y''y'$, we get $y'y'' = y''y'$. The latter implies that y' and y'' are powers of some full word z. We obtain x' and x'' are powers of z. We conclude that z_1, z_2', z_3 are contained in powers of z, implying that $w = z_1z_2'z_3$ is not primitive. □

Theorem 6. *Let ww', vv', uu' be three squares starting at the same position (not necessarily last occurrences) in a partial word with one hole, such that $|w| < |v| < 2|w| \leq |u| < |w| + |v|$. If the hole is not in ww', then $(w \wedge w')$ is not primitive.*

Proof. Let us assume that $(w \wedge w')$ is primitive, or that both w and w' are primitive. Since $|w| < |v| < 2|w| \leq |u| < |w| + |v|$, let us denote $v = wz_1$, $ww' = vz_2$ for some nonempty partial words z_1 and z_2, $u = ww'z_3$ for some partial word z_3, and let z_4 be a nonempty partial word such that z_4 is a prefix of u' and $|uz_4| = |w| + |v|$. We have $w' = z_1z_2$, $w = z_1'z_2'$, $v = z_1'z_2'z_1$ and $u = z_1'z_2'z_1z_2z_3$, where $z_i' \uparrow z_i$ for $i \in \{1, 2\}$. It follows easily that $|z_3z_4| = |z_1|$. Since $v \uparrow v'$, we get that $z_1'z_2' \uparrow z_2z_3z_4$, and by looking at the prefixes of length

$|w|$ of u and u', there exists a partial word z_5, with $|z_5| = |z_2z_3|$, such that z_4z_5 is a prefix of u' and $z_1'z_2' \uparrow z_4z_5$. Let z_1'' be such that z_4z_1'' is a prefix of u' of length $|z_4z_1|$. Since $v \uparrow v'$, $z_1'z_2'z_1 \uparrow z_2z_3z_4z_1''$, and by simplification $z_1 \uparrow z_1''$.

Assume that the hole is not in $ww' = z_1'z_2'z_1z_2$. Here $z_1'' \subset z_1 = z_1', z_2 = z_2'$. Since $v \uparrow v'$, $z_1z_2 \uparrow z_2z_3z_4$, which implies that $z_1z_2z_3z_4$ is strongly $|z_1|$-periodic by Lemma 3. Since $u \uparrow u'$, $z_1z_2 \uparrow z_4z_5$. The compatibilities $z_1z_2 \uparrow z_2z_3z_4$, $z_1z_2 \uparrow z_4z_5$ along with the fact that z_1z_2 is full imply that $z_2z_3z_4 \subset z_1z_2$, $z_4z_5 \subset z_1z_2$, and so $z_2z_3z_4 \uparrow z_4z_5$. Using Lemma 2, we get that $z_2z_3z_4z_5$ is weakly $|z_5|$-periodic.

If $z_3 = \varepsilon$, then $|z_4| = |z_1|$ and $|z_5| = |z_2|$. Here z_2z_4 is strongly $|z_1|$- and weakly $|z_5|$-periodic, and by Theorem 3, z_2z_4 is strongly $\gcd(|z_2|, |z_4|)$-periodic. Let x be the prefix of z_2 of length $\gcd(|z_2|, |z_4|)$. We get $z_2 = x^n$ and $z_4 \subset x^m$ for some positive integers n, m. If the hole is not in z_4, since $z_1z_2z_4$ is strongly $|x^m|$-periodic, we obtain $z_1 = x^m$. We conclude that $w = z_1z_2$ is not primitive, a contradiction. If the hole is in z_4, since $z_1z_2 \uparrow z_4z_5$ and $|z_2| = |z_5|$, we get $z_4 \subset z_1$ and $z_5 = z_2 = x^n$. Set $z_4 = x^{m_1}x_1'x^{m_2}$ and $z_1 = x^{m_1}x_1x^{m_2}$ where $x_1' \subset x$, $x_1' \subset x_1$, and $m_1 + m_2 + 1 = m$. Since $z_1z_2 \uparrow z_2z_4$, we get $x^{m_1}x_1x^{m_2}x^n \uparrow x^nx^{m_1}x_1'x^{m_2}$. By simplification, $x_1x^n \uparrow x^nx_1'$, and we get $x_1 = x$. Hence, $w = z_1z_2 = x^{m+n}$ is not primitive, a contradiction.

If $z_3 \neq \varepsilon$, then since $v' \uparrow v$, $z_2z_3z_4z_1'' \uparrow z_1z_2z_1$. By weakening, $z_2z_3z_4z_1'' \uparrow z_1''z_2z_1$, and by Lemma 2, $z_2z_3z_4z_1''z_2z_1$ is weakly $|z_1z_2|$-periodic. If $|z_5| = |z_1|$, then $z_5 = z_1''$. Also $|z_2| = |z_4|$. We have that both $z_2z_3z_4z_1''$ is weakly $|z_1|$- and weakly $|z_1z_2|$-periodic. By Theorem 3, $z_2z_3z_4z_1''$ is strongly $\gcd(|z_1|, |z_1z_2|)$-periodic. Let x be a full word of length $\gcd(|z_1|, |z_1z_2|) = \gcd(|z_2|, |z_1z_2|)$ such that $z_2 = x^n$, $z_3z_4 \subset x^m$, and $z_1'' \subset x^m$, for positive integers n, m. If the hole is neither in z_3 nor in z_4, then $w = z_1z_2 = z_2z_3z_4 \subset x^{m+n}$ is not primitive. If the hole is in z_3 or z_4, then $z_1'' = z_1 = x^m$, and so $w = z_1z_2 = x^{m+n}$ is not primitive.

If $|z_5| < |z_1|$, then z_5 is a prefix of z_1''. We have that $z_2z_3z_4z_5$ is weakly $|z_5|$-periodic, and $z_2z_3z_4z_1''z_2z_1$ is weakly $|z_1z_2|$-periodic. The latter implies that $z_2z_3z_4z_1''$, and hence $z_2z_3z_4z_5$, are weakly $|z_1z_2|$-periodic. By Theorem 3, $z_2z_3z_4z_5$ is strongly $\gcd(|z_5|, |z_1z_2|)$-periodic. Hence, there exists a full word x of length $\gcd(|z_5|, |z_1z_2|)$ such that both $z_2z_3z_4$ and z_5 are contained in powers of x. Since $|z_5| = |z_2z_3|$, both z_2z_3 and z_4 are contained in powers of x. If the hole is neither in z_3 nor in z_4, then $w = z_1z_2 = z_2z_3z_4$ is not primitive. If the hole is in z_3, then $w = z_1z_2 = z_4z_5$ is not primitive. If the hole is in z_4, then $z_2z_3 = z_5 = x^n$ and $z_4 \subset x^m$ for some positive integers n, m. Set $z_2 = (x'x'')^{n_1}x'$ and $z_3 = x''(x'x'')^{n_2}$, where $x = x'x''$ and where n_1, n_2 are integers satisfying $n_1 + n_2 + 1 = n$. Also set $z_4 = x^{m_1}x_1'x_2'x^{m_2}$, where the hole is in x_1' or x_2' and where m_1, m_2 are integers satisfying $m_1 + m_2 + 1 = m$. Since $z_1(x'x'')^{n_1}x' = z_1z_2 \uparrow z_4z_5 = x^{m_1}x_1'x_2'x^{m_2}x^n$, set $z_1 = x^{m_1}x_1x_2x^{m_2}x^{n_2}x_2''$, where x_1, x_2, x_2'' are such that $x_1' \subset x_1$, $x_1' \subset x'$, $x_2' \subset x_2$, $x_2' \subset x''$, and x_2'' is a prefix of x. Since $z_1z_2 \uparrow z_2z_3z_4$, we get $x^{m_1}x_1x_2x^{m_2}x^{n_2}x_2''(x'x'')^{n_1}x' \uparrow x^nx^{m_1}x_1'x_2'x^{m_2}$. By simplification, $x_1 = x'$ and $x_2 = x''$, and $z_1 = x^{n_2+m}x_2''$. Again, using simplification, $x^{m_2}x_2''(x'x'')^{n_1}x' \uparrow x^{n_1}x_1'x_2'x^{m_2}$.

If $n_1 > 0$ and $m_2 > 0$, then $x'x'' = x_2''x' = x''x'$. In this case, x', x'' are powers of a common word, and so $w = z_1z_2$ is not primitive, a contradiction.

The case where $n_1 > 1$ and $m_2 = 0$ similarly follows. If $n_1 = 1$ and $m_2 = 0$, then $x_2''x'x''x' \uparrow x'x''x_1'x_2'$. If the hole is in x_1', then $x_2' = x''$. Here $x'x'' = x_2''x'$ and $x_1'x'' \uparrow x''x'$. The latter implies the existence of words y', y'' such that $x_1' \subset y'y''$, $x' = y''y'$ and $x'' = (y'y'')^r y'$ for some integer $r \geq 0$. By Lemma 1, since $x_1' \subset y'y''$ and $x_1' \subset x' = y''y'$, we obtain $y'y'' = y''y'$ and so y', y'' are powers of a common word. Since $x'x'' = x_2''x'$, x_2'' is also a power of that word, and thus $x_2'' = x''$ and the result follows as above. The proof when the hole is in x_2' is similar. If $n_1 = 0$, then since $z_1z_2 \uparrow z_4z_5$, we get $x^{n_2+m}x_2''x' = z_1z_2 \uparrow z_4z_5 = x^{m_1}x_1'x_2'x^{m_2}x^n$. Looking at the suffixes of z_1z_2 and z_4z_5 of length $|x|$, we get $x_2''x' = x'x''$. We get $w = z_1z_2 = x^{n_2+m}x_2''x' = x^{n_2+m}x'x'' = x^{n_2+m+1} = x^{n_2+m_1+m_2+2}$ is not primitive.

If $|z_5| > |z_1|$, then z_1'' is a prefix of z_5. Set $z_5 = z_{1,1}z_{1,2}\cdots z_{1,s}z_6$ for some nonempty partial words $z_{1,1}, z_{1,2}, \ldots, z_{1,s}, z_6$, where $z_{1,1} = z_1''$, $|z_{1,1}| = |z_{1,2}| = \cdots = |z_{1,s}| = |z_1| \geq |z_6|$, and where $s \geq 1$ is an integer. Note that it is possible that one of $z_{1,1}, \ldots, z_{1,s}, z_6$ contains the hole. Since $|z_2z_3| = |z_5| = s|z_1| + |z_6| = (s-1)|z_1| + |z_3z_4| + |z_6|$, we get $|z_2| = (s-1)|z_1| + |z_4z_6|$. The compatibility $z_1z_2 \uparrow z_2z_3z_4$ implies that $z_2 = z_1^{s-1}z_6'z_4'$ for some z_4', z_6' such that $z_4 \subset z_4'$ and $|z_6'| = |z_6|$. We obtain $z_1z_1^{s-1}z_6'z_4' = z_1z_2 \uparrow z_2z_3z_4 = z_1^{s-1}z_6'z_4'z_3z_4$, and by simplification $z_1z_6' \uparrow z_6'z_4'z_3$. We deduce that z_6' is a prefix of z_1.

Since $z_2z_3z_4 \uparrow z_4z_5 = z_4z_{1,1}z_{1,2}\cdots z_{1,s}z_6$, let z_4'' be the prefix of z_2 such that $z_4 \subset z_4''$ (note that z_4'' is also a prefix of z_1 since $z_1z_2 \uparrow z_2z_3z_4$). Let $z_{1,1}', z_{1,2}', \ldots, z_{1,s-1}', z_6''$ be such that $z_2 = z_4''z_{1,1}'z_{1,2}'\cdots z_{1,s-1}'z_6''$, where $z_{1,i}' \uparrow z_{1,i}$ for $1 \leq i < s$, and $z_6''z_3z_4 \uparrow z_{1,s}z_6$. Since

$$z_1z_4''z_{1,1}'z_{1,2}'\cdots z_{1,s-1}'z_6'' = z_1z_2 \uparrow z_4z_5 = z_4z_{1,1}z_{1,2}\cdots z_{1,s}z_6$$

we get, using simplification, $z_1z_4'' \uparrow z_4z_{1,1} = z_4z_1''$, $z_{1,i}' \uparrow z_{1,i+1}$ for $1 \leq i < s$, and $z_6'' \uparrow z_6$. The fact that $z_1z_4'' \uparrow z_4z_1''$ implies that z_4'' is compatible with a suffix of z_1''. Since $z_{1,i}'$ is full for every i, $z_{1,i} \uparrow z_{1,i+1}$ for $1 \leq i < s$.

Let us consider the cases when the hole is in z_3 or z_4 or z_5 (the proof when all of z_3, z_4 and z_5 are full is simpler). There are three cases to consider: Case 1 (the hole is not in $z_{1,i}$ for any $1 \leq i \leq s$), Case 2 (the hole is in $z_{1,1}$), Case 3 (the hole is in $z_{1,i}$ for some $1 < i \leq s$). We treat Cases 1 and 3.

Case 1. The hole is not in $z_{1,i}$ for any $1 \leq i \leq s$.

Here, the hole is in z_3 or z_4 or z_6. Moreover, $z_1 = z_{1,1} = z_{1,1}' = z_{1,2} = z_{1,2}' = \cdots = z_{1,s-1} = z_{1,s-1}' = z_{1,s}$, and so $z_5 = z_1^s z_6$ and $z_2 = z_1^{s-1}z_6'z_4' = z_4''z_1^{s-1}z_6''$. We deduce that $z_6'z_4' = z_4''z_6''$.

If the hole is not in z_4, then $z_4 = z_4' = z_4''$ and so $z_6'z_4 = z_4z_6''$. There exist y', y'' such that $z_6' = y'y''$, $z_6'' = y''y'$ and $z_4 = (y'y'')^r y'$ for some integer $r \geq 0$. The prefix of length $|z_6|$ of z_1 is $z_6'' = y''y'$ (recall that $z_6''z_3z_4 \uparrow z_1z_6$). Since z_6' is a prefix of z_1, the prefix of length $|z_6|$ of z_1 is $z_6' = y'y''$. By Lemma 1, we obtain $z_6' = y'y'' = y''y' = z_6''$. We have that $z_2z_3z_4z_5 = z_2z_3z_4z_1^s z_6$ is weakly $|z_5|$-periodic, and $z_2z_3z_4z_1z_2z_1 = z_2z_3z_4z_1z_1^{s-1}z_6'z_4z_1$, and hence $z_2z_3z_4z_1^s z_6'$ and $z_2z_3z_4z_1^s z_6''$ and $z_2z_3z_4z_1^s z_6$, are weakly $|z_1z_2|$-periodic. By Theorem 3, $z_2z_3z_4z_1^s z_6$ is strongly $\gcd(|z_5|, |z_1z_2|)$-periodic. There exists a full word x of length $\gcd(|z_5|, |z_1z_2|)$ such that both $z_2z_3z_4$ and $z_5 = z_1^s z_6$ are contained in powers of x. Since $|z_5| = |z_2z_3|$, both z_2z_3 and z_4 are contained in powers of x. If the hole is in z_3, then $w = z_1z_2 = z_4z_5$ is not primitive. If the hole is in z_6, then $w = z_1z_2 = z_2z_3z_4$ is not primitive.

If the hole is in z_4, then $z_4 \subset z_4'$, $z_4 \subset z_4''$, and $z_6 = z_6''$. Since $z_1z_6' \uparrow z_6'z_4'z_3$, it follows that $z_1z_6'z_4'z_3$ is strongly $|z_1|$-periodic. Also, since $z_1z_6 = z_{1,s}z_6 \uparrow z_6''z_3z_4 = z_6z_3z_4$, we have that $z_1z_6z_3z_4$ is strongly $|z_1|$-periodic. Then both z_6 and z_6' are prefixes of z_1, and since $|z_6| = |z_6'|$, we get that $z_6 = z_6'$. Hence, because $z_6z_4'z_3 = z_1z_6 \uparrow z_6z_3z_4$, it follows that $z_4'z_3 \uparrow z_3z_4$. From Lemmas 3 and 1, there exists a word y such that $z_4 \subset y^{n_1} = z_4'$ and $z_3 = y^{n_2}$, for some integers n_1, n_2.

Moreover, since $|z_1| = |z_4'z_3|$ and $|z_6| \leq |z_1|$, and $z_1z_6z_4'z_3$ is $|z_1|$-periodic, there exist words y', y'' such that $z_1 = (y'y'')^{n_1+n_2}$, $z_6 = (y'y'')^{n_3}y'$, $z_4'z_3 = (y''y')^{n_1+n_2}$ and $y = y''y'$. We get that

$$z_2 = ((y'y'')^{n_1+n_2})^{s-1}(y'y'')^{n_1+n_3}y', z_5 = ((y'y'')^{n_1+n_2})^s(y'y'')^{n_3}y'$$

But, from $u' \uparrow u$, the prefixes of length $|w|$ are compatible. Hence,

$$z_4((y'y'')^{n_1+n_2})^s(y'y'')^{n_3}y' = z_4z_5 \uparrow z_1z_2 = ((y'y'')^{n_1+n_2})^s(y'y'')^{n_1+n_3}y'$$

It follows that $z_4 \subset (y'y'')^{n_1}$. But, $z_4 \subset y^{n_1} = (y''y')^{n_1}$, gives us that the prefix of length $|y|$ of z_4 is contained in both $y'y''$ and $y''y'$, and so y' and y'' are powers of a common word. Thus $w = z_1z_2 = ((y'y'')^{n_1+n_2})^s(y'y'')^{n_1+n_3}y'$ is also a power of that common word, which is a contradiction.

Case 3. The hole is in $z_{1,i}$ for some $1 < i \leq s$.

Here, $z_4 = z_4' = z_4''$ and $z_6'' = z_6$. Moreover, $z_1 = z_1'' = z_{1,1} = z_{1,1}' = z_{1,2} = z_{1,2}' = \cdots = z_{1,i-1} = z_{1,i-1}' \supset z_{1,i}$, and $z_{1,i} \subset z_{1,i}' = z_{1,i+1} = z_{1,i+1}' = \cdots = z_{1,s-1} = z_{1,s-1}' = z_{1,s}$. Using the facts that $z_2 = z_1^{s-1}z_6'z_4$ and $z_1z_6' = z_6'z_4z_3$, we deduce that $z_2 = z_1^{s-1}z_6'z_4 = z_1^{s-2}z_1z_6'z_4 = z_1^{s-2}z_6'z_4z_3z_4 = z_1^{s-3}z_1z_6'z_4z_3z_4 = \cdots = z_6'z_4(z_3z_4)^{s-1}$. Note that $z_6'z_4$ is both a prefix and a suffix of z_2. Since z_4 is a prefix of z_2 and z_6 is a suffix of z_2, we get $z_6'z_4 = z_4z_6$ which implies the existence of words y', y'' such that $z_6' = y'y''$, $z_6 = y''y'$, and $z_4 = (y'y'')^r y'$ for some integer $r \geq 0$.

Set $z_1 = z_4z_3'$ for some z_3'. Here $z_2 = z_1 \cdots z_1z_6'z_4 = z_4z_{1,1}' \cdots z_{1,s-1}'z_6$ implies that $z_{1,1}' = \cdots = z_{1,s-2}' = z_3'z_4$, and $z_1z_4z_6 = z_1z_6'z_4 = z_4z_{1,s-1}'z_6$. The latter implies that $z_{1,s-1}' = z_3'z_4$. If $i < s$, then since $z_{1,i} \subset z_{1,i}' = z_3'z_4$ and $z_{1,i} \subset z_1 = z_4z_3'$, we get $z_3'z_4 = z_4z_3'$ by Lemma 1. If $i = s$, then $z_3'z_4 = z_{1,s-1}' = z_1 = z_4z_3'$. In either case, $z_5 = z_1^{i-1}z_{1,i}z_{1,s}^{s-i}z_6 = z_1^{i-1}z_{1,i}z_1^{s-i}z_6$ and $z_2 = z_1^{s-1}z_6'z_4 = z_4z_1^{s-1}z_6$.

The prefix of length $|z_6|$ of $z_{1,s}$ is contained in $z_6 = y''y'$ and so the prefix of length $|z_6|$ of $z_{1,i}$ is contained in $y''y'$ (recall that $z_6z_3z_4 \uparrow z_{1,s}z_6$). Since z_6' is a prefix of z_1, the prefix of length $|z_6|$ of z_1 is $y'y''$ and so the prefix of length $|z_6|$ of $z_{1,i}$ is contained in $y'y''$. By Lemma 1, $z_6' = y'y'' = y''y' = z_6$. We have that $z_2z_3z_4z_5 = z_2z_3z_4z_1^{i-1}z_{1,i}z_1^{s-i}z_6$ is weakly $|z_5|$-periodic, and $z_2z_3z_4z_1z_2z_1 = z_2z_3z_4z_1z_1^{s-1}z_6'z_4z_1$, and hence $z_2z_3z_4z_1^sz_6'$, $z_2z_3z_4z_1^sz_6$ and $z_2z_3z_4z_1^{i-1}z_{1,i}z_1^{s-i}z_6$, are weakly $|z_1z_2|$-periodic. By Theorem 3, we have that $z_2z_3z_4z_1^{i-1}z_{1,i}z_1^{s-i}z_6$

is strongly $\gcd(|z_5|, |z_1 z_2|)$-periodic, and there exists a full word x of length $\gcd(|z_5|, |z_1 z_2|)$ such that $z_2 z_3 z_4$ is a power of x and $z_5 = z_1^{i-1} z_{1,i} z_1^{s-i} z_6$ is contained in a power of x. Since $|z_5| < |z_2 z_3 z_4|$, it must be the case that $w = z_2 z_3 z_4 = x^p$ for some $p \geq 2$, a contradiction with the fact that w is primitive. □

Note that Theorem 6 does not necessarily hold when the hole is in the shortest square. Consider for example, the partial word with one hole

$$aba\diamond ababaaabab\underline{a}babab\underline{\underline{a}}baaababaaab\overline{a}babababaaababababababaaababaaab$$

that has three squares, ww', vv', uu', starting at position zero:

$w \ = aba\diamond ababaaabab$
$w' = ababababaaabab$
$v \ \ = aba\diamond ababaaabababababab$
$v' \ = abaaababaaabababababab$
$u \ \ = aba\diamond ababaaababababababababaaababaaab$
$u' \ = ababababaaababababababaaababaaab$

of length 14×2, 20×2 and 32×2 respectively (we have underlined with one line the first letter of w', with two lines the first letter of v', and have overlined the first letter of u'). Note that the condition $|w| < |v| < 2|w| \leq |u| < |w| + |v|$ holds. Here, the hole is in ww' and $(w \wedge w') = w$ is primitive (actually, all six words w, w', v, v', u and u' are primitive).

4 Conclusion

In this paper, we have proved the following extension of the three-squares theorem to partial words with one hole.

Theorem 7. *Let ww', vv', uu' be three squares starting at the same position (not necessarily last occurrences) in a partial word with one hole, such that $|w| < |v| < |u|$ and $(w \wedge w')$ is primitive. Then $|u| \geq 2|w|$. Moreover, the following hold:*

- *If $|v| \geq 2|w|$, then $|w| + |v| \leq |u|$.*
- *If $|v| < 2|w|$, then $|w| + |v| \leq |u|$ or the hole is in ww'.*

Proof. By Theorem 5, $|u| \geq 2|w|$. By Theorem 4, if $|v| \geq 2|w|$, then $|w|+|v| \leq |u|$. By Theorem 6, if $|v| < 2|w|$, then $|w| + |v| \leq |u|$ or the hole is in ww'. □

Corollary 1. *If ww', vv', uu' are three squares starting at the same position (not necessarily last occurrences) in a full word, such that $|w| < |v| < |u|$ and w is primitive, then $|w| + |v| \leq |u|$.*

Note that the three-squares theorem does not hold for partial words with two holes since, for example, the partial word $a\diamond baabbaabaab\diamond abbaab$ has three squares starting at position zero:

$$(abba)^2,\ (aabaabb)^2, \text{ and } (aabaabbaab)^2$$

of length 4×2, 7×2 and 10×2 respectively.

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