Counting maximal-exponent factors in words

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joint work with Golnaz Badkobeh and Maxime Crochemore

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What is it?!

tin

What is it?!

tin^2

What is it?!

$$tin^2 = tintin$$

What is it?!

$$tin^3 = tintintin$$

What is it?!

$$tin^3 = tintintin$$

$$an$$

What is it?!

$$tin^3 = tintintin$$

$$an^{\frac{3}{2}} = ana$$

What is it?!

$$tin^3 = tintintin$$

$$an^{\frac{5}{2}} = anana$$

What is it?!

$$tin^3 = tintintin$$

$$b(an)^{\frac{5}{2}} = banana$$

What is it?!

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$$b(an)^{\frac{5}{2}} = banana$$

$$mississippi$$

What is it?!

$$tin^3 = tintintin$$

$$b(an)^{\frac{5}{2}} = banana$$

$$mississippi = m(iss)^{\frac{7}{3}}ppi$$

Motivation

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abbaabbaab and ababa represent MEFs with $u =$ ab and $v =$ ba, or $u =$ a and $v =$ b, respectively, where the exponent of the MEF is 2.5

Previous Results

Lemma (Badkobeh, Crochemore)

Consider two occurrences of MEFs with the same border length b starting at respective i and j positions in the word. Then, $|j - i| > b$.

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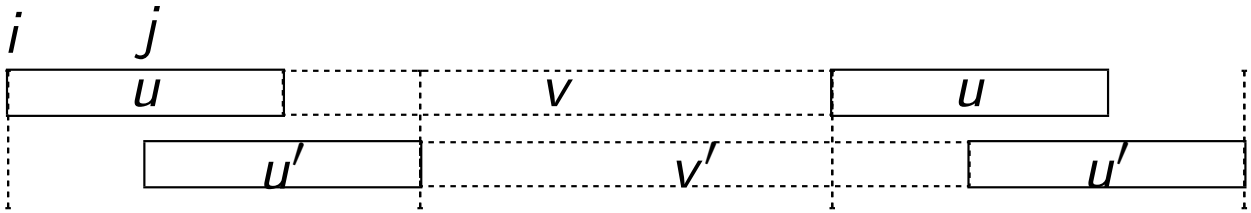

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A MEF uvu is a δ-MEF if its border length $b = |u| = |uvu| - p(uvu)$ satisfies $2\delta < b \leq 4\delta$. Then any MEF is a δ-MEF for some $\delta \in \Delta$, where $\Delta = \{1/4, 1/2, 1, 2, 2^2, 2^3, \ldots\}$.

Lemma (Badkobeh, Crochemore)

Let uvu and $u'v'u'$ be occurrences of δ-MEFs in w whose left borders mid-positions are at respective positions i and j on w. Then, $|j - i| \geq \delta$.

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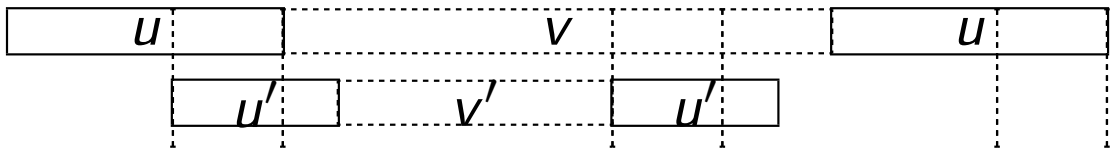

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The direct consequence of the previous lemma is that if uvu and $u'v'u'$ are two δ-MEFs, then u cannot contain u'. Hence, we get the upper bound:

$$\left(\sum_{b=1}^{b=k} \frac{n}{b+1}\right) + \frac{1}{k}\left(2 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) n = \left(\sum_{b=1}^{b=k} \frac{n}{b+1}\right) + \frac{4n}{k}.$$

Small tweak

Lemma

Let $S = [r, .., s]$ be an interval of integers such that $r > \frac{2s}{3}$. Then within every $r+1$ positions, there are at most two MEFs with border lengths in S.

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We introduce the notion of γ-MEFs, for a positive real number γ: a MEF uvu is a γ-MEF if its border length $b = |u|$ satisfies $2\gamma \leq b < 3\gamma$. Then any MEF is a γ-MEF for some $\gamma \in \Gamma$ where $\Gamma = \{\frac{1}{2}, \frac{1}{2} \cdot (\frac{3}{2}), \frac{1}{2} \cdot (\frac{3}{2})^2, \ldots\}$.

Corollary

For three consecutive γ-MEFs starting at positions i, j and k, respectively, $\max\{k-i, j-i\} > 3\gamma$.

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Theorem

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Corollary

There are less than $n/2$ occurrences of MEFs with border length at least 8 in a word of length n.

Double borders

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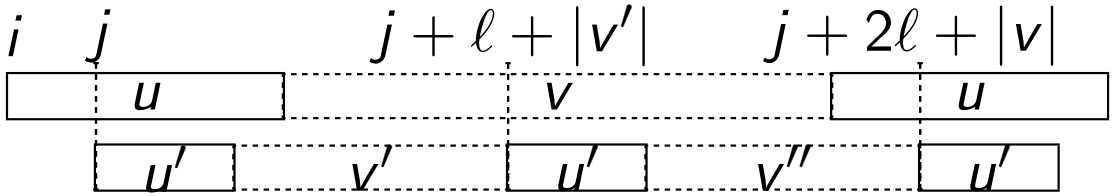

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Proposition

There are at most $2n/(2\ell+1)$ *MEFs with border lengths* ℓ *and* 2ℓ *in a word of length* n.

Exponential borders

Example

Consider words $u = ab$, $v = ac$ and alphabet $\Sigma = \{a_1, b_1, a_2, b_2, \ldots\}$ for which $a, b, c \notin \Sigma$. Let $S_1 = ua_1vb_1$ and $S_i = S_{i-1}ua_ivb_i$ for $i \geq 2$.

$$ab \cdot ac \cdot ab \cdot ac \cdot ab \cdot ac \cdot ab \cdot ac \cdot ab \cdot ac \cdot ab$$

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Lemma

Every word of length n contains, for any positive integer $\ell \leq \frac{n}{2}$, at most $\frac{n^2}{\ell n + 2\ell}$ MEFs with the border length of the form $\ell\, 2^i$, for any $0 \leq i \leq \log(\frac{n}{\ell})$.

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Lemma

There are at most $13n/10$ occurrences of MEFs whose border length is at most 7 in a word of length n.

Main Result

Theorem

There exist at most $1.8n$ *number of occurrences of MEFs in a word of length* n.

Special cases

Lemma

There are at most n occurrences of maximal-exponent factors in a word of length n, whenever the maximal exponent is greater than 1.5*.*

Theorem

Every length n word contains at most n occurrences of MEFs whenever the length of these factors is not a multiple of their longest border.

Lower bound – $5n/6$

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Construct the infinite word $\Omega = \prod_{i=1}^{\infty} \left(\prod_{j=1}^{8} u_{(j,i)} \right)$, from the sequence:

$$u_{(1,i)} = \mathtt{a}_1\ \mathtt{b}_1\ \mathtt{c}_1\ \mathtt{a}_2\ \mathtt{d}_1\ \mathtt{a}_3\ \mathtt{b}_2\ \mathtt{e}_1\ \mathtt{f}_{1,i}$$
$$u_{(2,i)} = \mathtt{a}_1\ \mathtt{b}_3\ \mathtt{c}_2\ \mathtt{a}_2\ \mathtt{d}_2\ \mathtt{a}_3\ \mathtt{b}_4\ \mathtt{e}_1\ \mathtt{f}_{2,i}$$
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$$u_{(5,i)} = \mathtt{a}_1\ \mathtt{b}_1\ \mathtt{c}_1\ \mathtt{a}_2\ \mathtt{d}_5\ \mathtt{a}_3\ \mathtt{b}_2\ \mathtt{e}_3\ \mathtt{f}_{5,i}$$
$$u_{(6,i)} = \mathtt{a}_1\ \mathtt{b}_3\ \mathtt{c}_2\ \mathtt{a}_2\ \mathtt{d}_6\ \mathtt{a}_3\ \mathtt{b}_4\ \mathtt{e}_3\ \mathtt{f}_{6,i}$$
$$u_{(7,i)} = \mathtt{a}_1\ \mathtt{b}_1\ \mathtt{c}_3\ \mathtt{a}_2\ \mathtt{d}_7\ \mathtt{a}_3\ \mathtt{b}_2\ \mathtt{e}_4\ \mathtt{f}_{7,i}$$
$$u_{(8,i)} = \mathtt{a}_1\ \mathtt{b}_3\ \mathtt{c}_4\ \mathtt{a}_2\ \mathtt{d}_8\ \mathtt{a}_3\ \mathtt{b}_4\ \mathtt{e}_4\ \mathtt{f}_{8,i}$$

Proposition

The ratio between the length of the prefixes of Ω and the number of occurrences of its maximal-exponent factors they contain tends to $5/6$.

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Note that the maximal exponent of factors in Ω is $10/9$ and that its construction can be extended to whatever exponent of the form $(2^\ell + 2)/(2^\ell + 1)$, in a similar fashion.

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Observe that letters $\mathtt{f}_{j,i}$ occurring in Ω can be drawn from an 11-letter alphabet disjoint from Σ.

Thanks

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